ECE 6775
High-Level Digital Design Automation
Fall 2024

More Scheduling

Cornell University

CSL

Announcements

- Lab 3 due Monday
- HW 2 will be released soon
  - Two problems related to pipelining need next week's lecture content

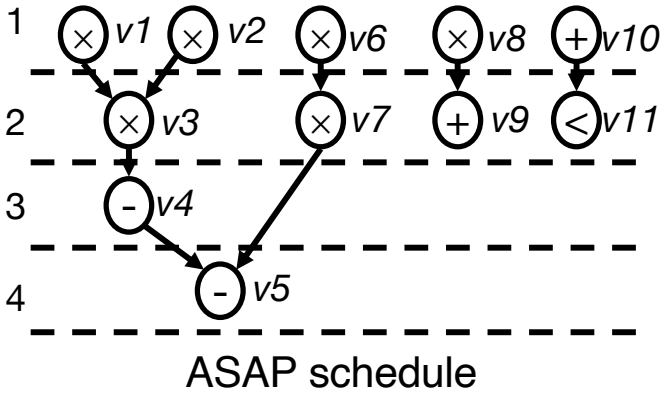
Agenda

- ILP for time-constrained scheduling
- Heuristic algorithms for constrained scheduling
  - List scheduling
  - SDC-based scheduling

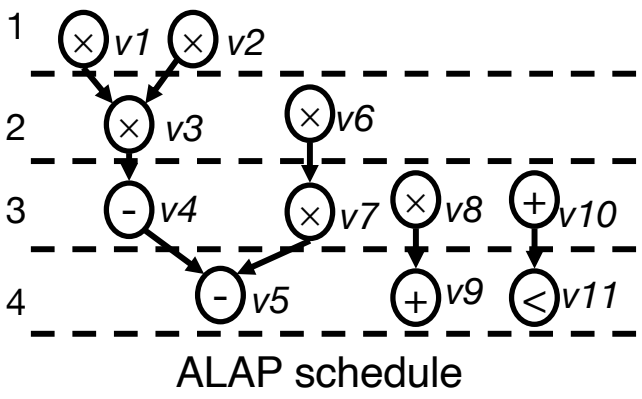
Recap: ILP Formulation for Dependence Constraints

- Using ASAP and ALAP, the non-trivial inequalities are: (assuming no chaining and single-cycle ops)

$$2x_{7,2} + 3x_{7,3} - x_{6,1} - 2x_{6,2} \geq 1$$
$$4x_{5,4} - 2x_{7,2} - 3x_{7,3} \geq 1$$
$$2x_{9,2} + 3x_{9,3} + 4x_{9,4} - x_{8,1} - 2x_{8,2} - 3x_{8,3} \geq 1$$
$$2x_{11,2} + 3x_{11,3} + 4x_{11,4} - x_{10,1} - 2x_{10,2} - 3x_{10,3} \geq 1$$



ASAP schedule



ALAP schedule

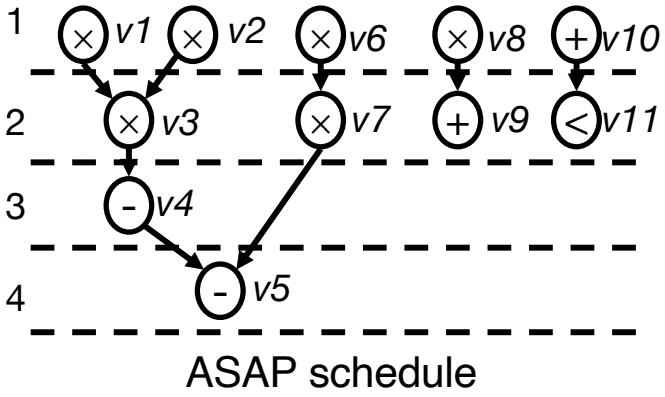
assume L=4 and no chaining

Recap: ILP Formulation for Resource Constraints

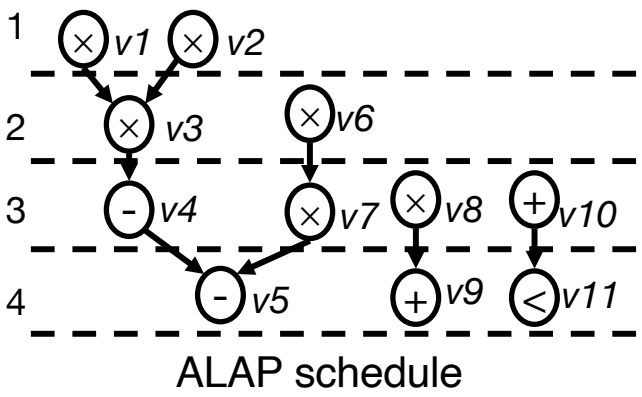
- Resource constraints (assuming 2 multipliers and 1 ALU)

$$x_{1,1} + x_{2,1} + x_{6,1} + x_{8,1} \leq 2$$
$$x_{3,2} + x_{6,2} + x_{7,2} + x_{8,2} \leq 2$$
$$x_{7,3} + x_{8,3} \leq 2$$

$$x_{10,1} \leq 1$$
$$x_{9,2} + x_{10,2} + x_{11,2} \leq 1$$
$$x_{4,3} + x_{9,3} + x_{10,3} + x_{11,3} \leq 1$$
$$x_{5,4} + x_{9,4} + x_{11,4} \leq 1$$



ASAP schedule



ALAP schedule

assume L=4 and no chaining

Another Exercise: Formulating ILP

- Minimize the number of classrooms that the school must allocate for the following courses
- Steps to formulate the ILP
  1. Create variables
  2. Each course to be scheduled to exactly one of the preferred slots
  3. Determine the number of rooms required (by creating derived variables)
  4. Set up the objective function

Course	Preferred Slots
A	(1) (2)
B	(1) (3)
C	(2) (3)
D	(2)

(1) 8:00 – 10:00am
(2) 10:00am – 12:00pm
(3) 12:00 – 2:00pm

① $x_{i,s}$: course i uses slot s

② $x_{A,1} + x_{A,2} = 1$
$x_{B,1} + x_{B,3} = 1$
$x_{C,2} + x_{C,3} = 1$
$x_{D,2} = 1$

③ $r_1 = x_{A,1} + x_{B,1}$
$r_2 = x_{A,2} + x_{C,2} + x_{D,2}$
$r_3 = x_{B,3} + x_{C,3}$

④ Objective: min max {r1, r2, r3}

Linearize

min R
$R \geq r_1, R \geq r_2, R \geq r_3$

Time-Constrained Scheduling (TCS)

- Dual problem of resource-constrained scheduling
  - Overall latency is given as a constraint (deadline)
  - Minimize the total cost in terms of area (or resource usage), power, etc.

- NP-hard problem
  - ILP formulation is exact but is not a polynomial-time solution
  - Force-directed scheduling is a well-known heuristic for TCS (see De Micheli chapter 5.4.4)

Example: ILP Formulation for TCS

- ILP for time-constrained scheduling

  minimize $c^T y$

  $x_{1,1} + x_{2,1} + x_{6,1} + x_{8,1} \leq y_1$

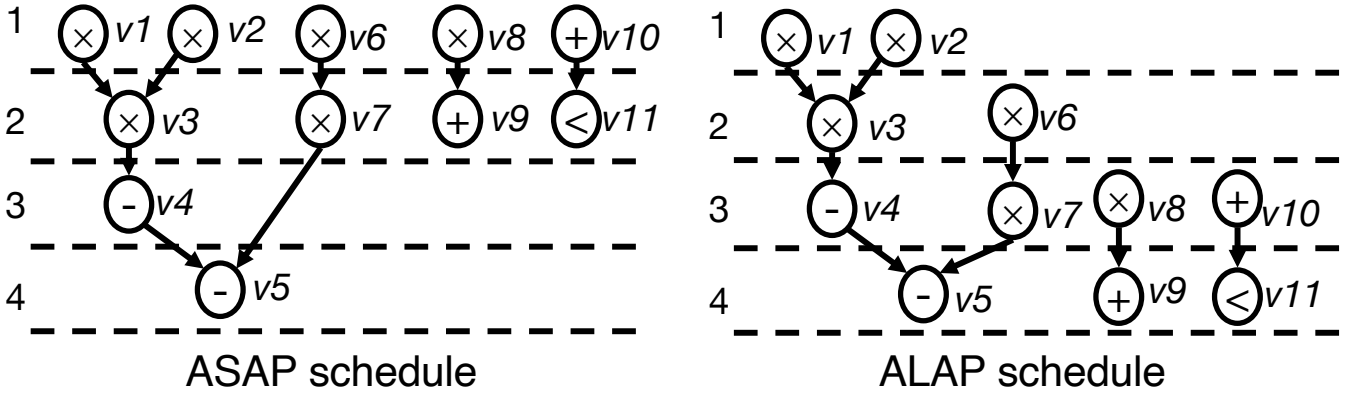
  $x_{3,2} + x_{6,2} + x_{7,2} + x_{8,2} \leq y_1$

  $x_{7,3} + x_{8,3} \leq y_1$

  $x_{5,4} + x_{9,4} + x_{11,4} \leq y_2$

  ...

  What does the *y* vector represent?



Constrained Scheduling in HLS

- Constrained scheduling
  - General case NP-hard
  - Resource-constrained scheduling (RCS)
    - Minimize latency given constraints on area or resources
  - Time-constrained scheduling (TCS)
    - Minimize resources subject to bound on latency

- Exact methods
  - Integer linear programming (ILP)
  - Symbolic scheduling using BDDs
  - Hu's algorithm for a very restricted problem

- Heuristics
  - List scheduling
  - Force-directed list scheduling
  - SDC-based scheduling

  …

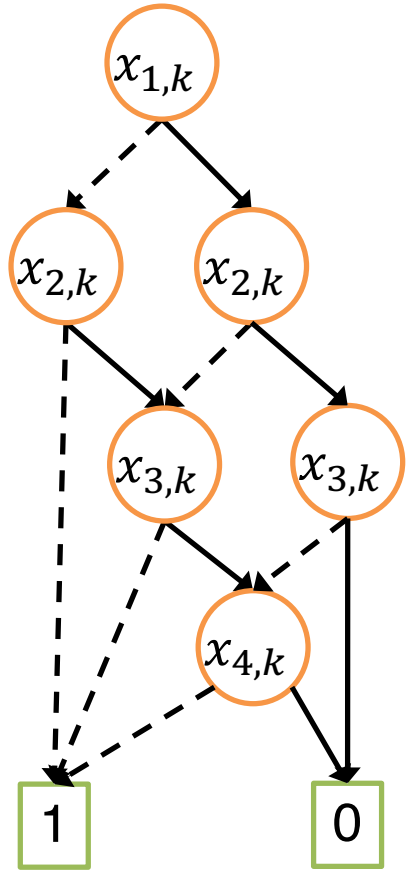
Symbolic Scheduling: Representing Resource Constraints with BDD (an example)

- Assume 2 multipliers are available and there are 4 potential multiply operations at control step *k*

- The following Boolean expression captures the resource constraint at step k

$$x'_{1,k} \cdot x'_{2,k} + x_{1,k}' \cdot x_{3,k}' + x_{1,k}' \cdot x_{4,k}' + x_{2,k}' \cdot x_{3,k}' + x_{2,k}' \cdot x_{4,k}' + x_{3,k}' \cdot x_{4,k}'$$

This expression indicates that at least (4 - 2) multiplication operations (among 4 potential operations in step k) cannot be scheduled to the same step



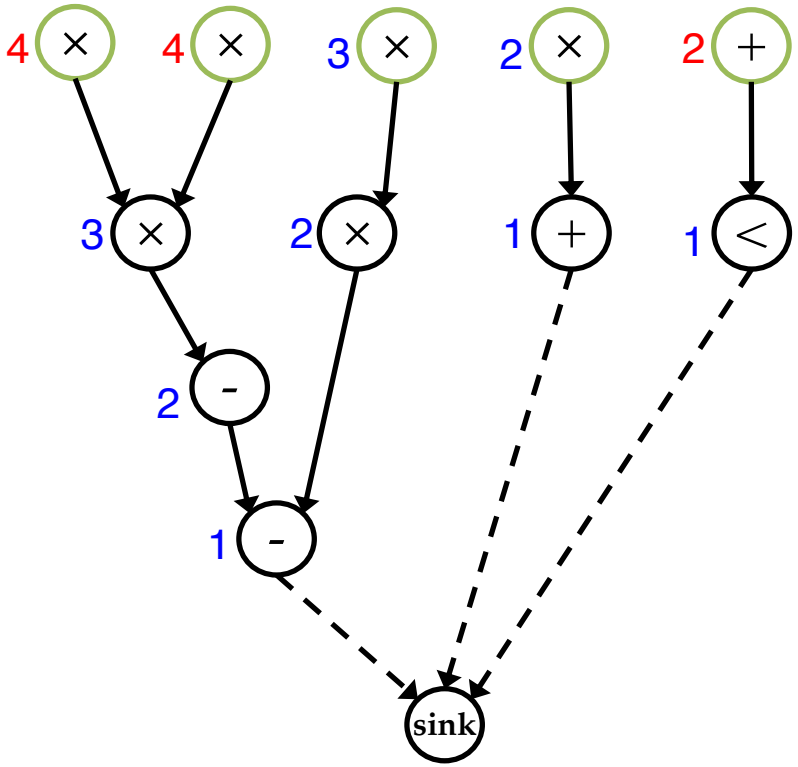
Radivojevic & Brewer, Symbolic Techniques for Optimal Scheduling, SASIMI'1993.

List Scheduling

- A widely-used heuristic algorithm for RCS
  - Schedule one control step (cycle) at a time
  - Maintain a list of “ready” operations considering dependence
  - Assign priorities to operations; most “critical” operations (with the highest priorities) go first

- Often refers to a family of algorithms
  - Typically classified by the way priority function is calculated
    - Static priority: Priorities are calculated once before scheduling
    - Dynamic priority calculation: Priorities are updated during scheduling

Static Priority Example: Node Height

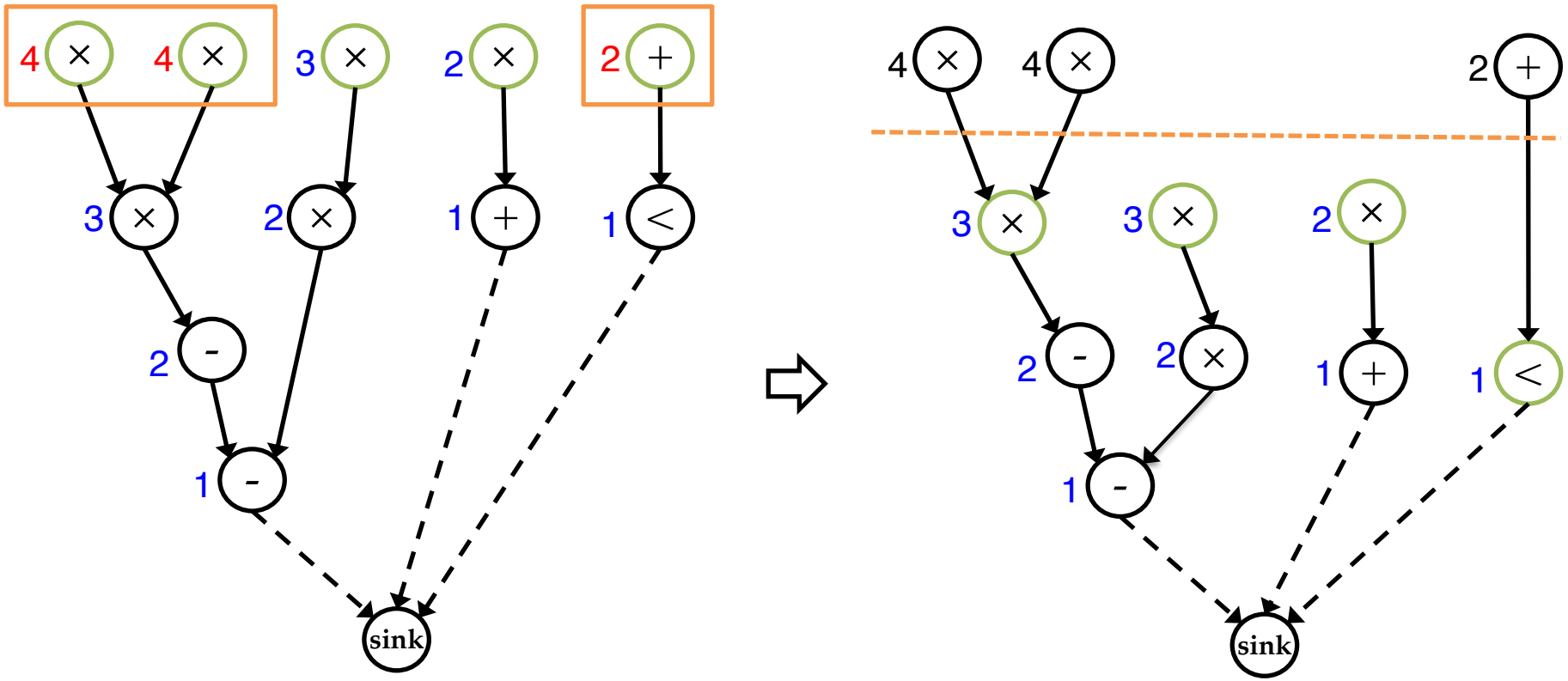


Nodes are labelled with distance to sink (height)

Ready operations are colored in green

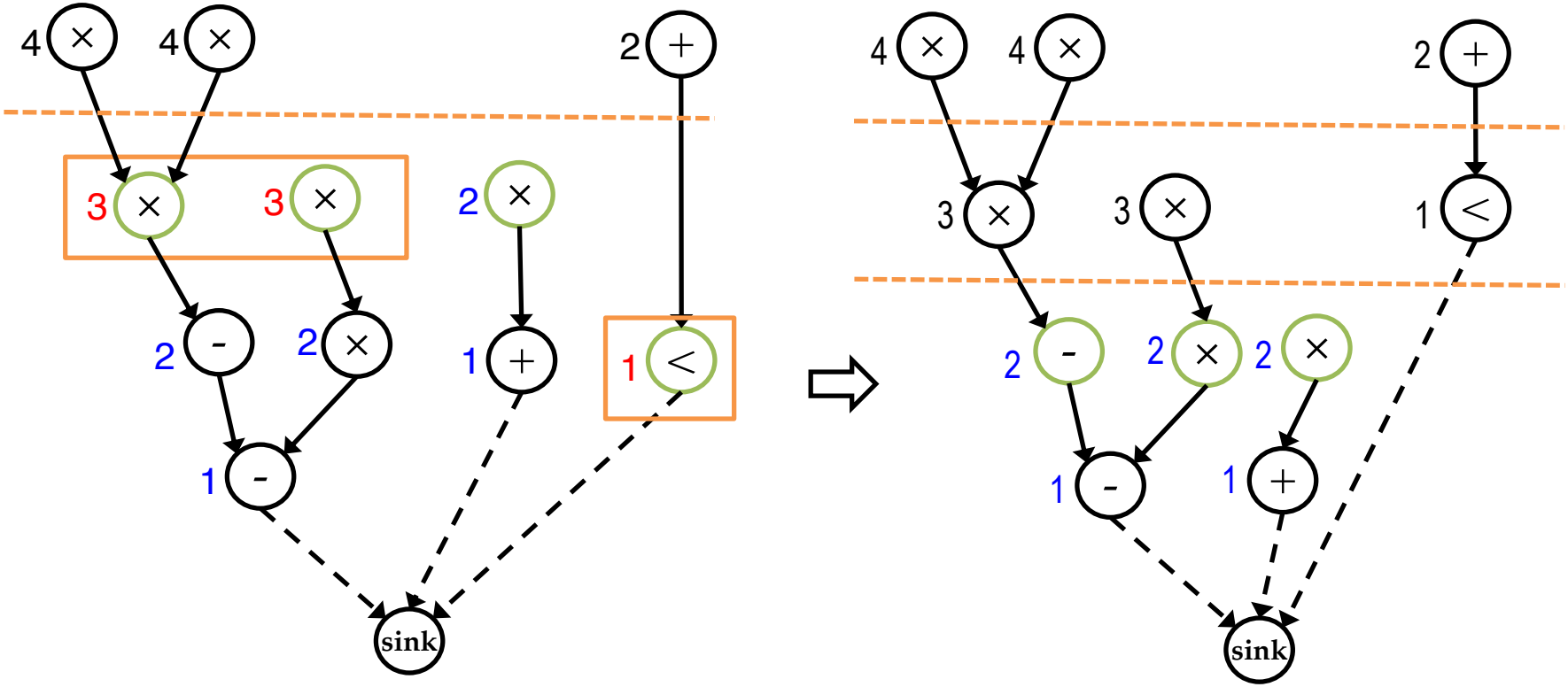
- Assumptions:
  - All operations have unit delay
  - 2 MULTs, 1 AddSub, and 1 CMP available

Ready Nodes with Highest Priorities Picked First



- Assumptions:
  - All operations have unit delay
  - 2 MULTs, 1 AddSub, and 1 CMP available

Update Ready Nodes and Repeat for Each Step



- Assumptions:
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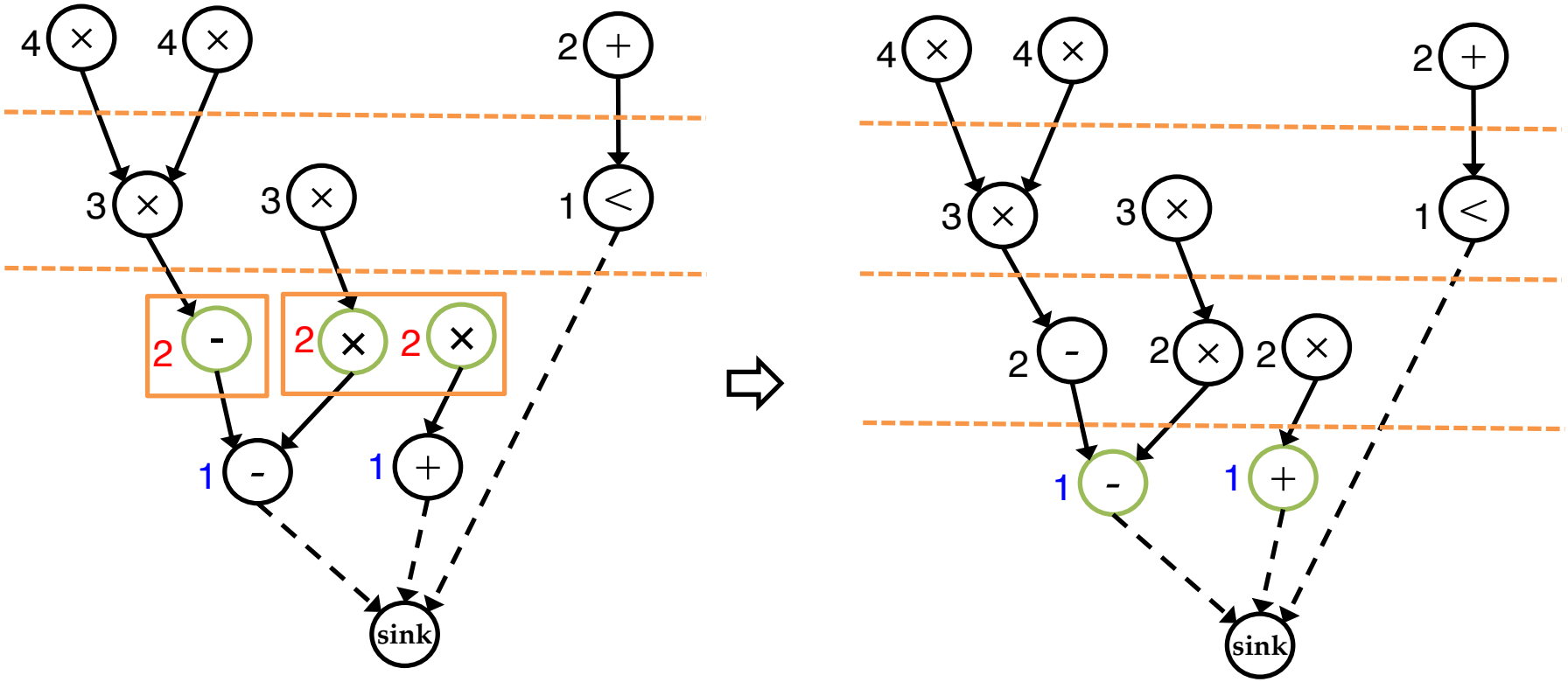
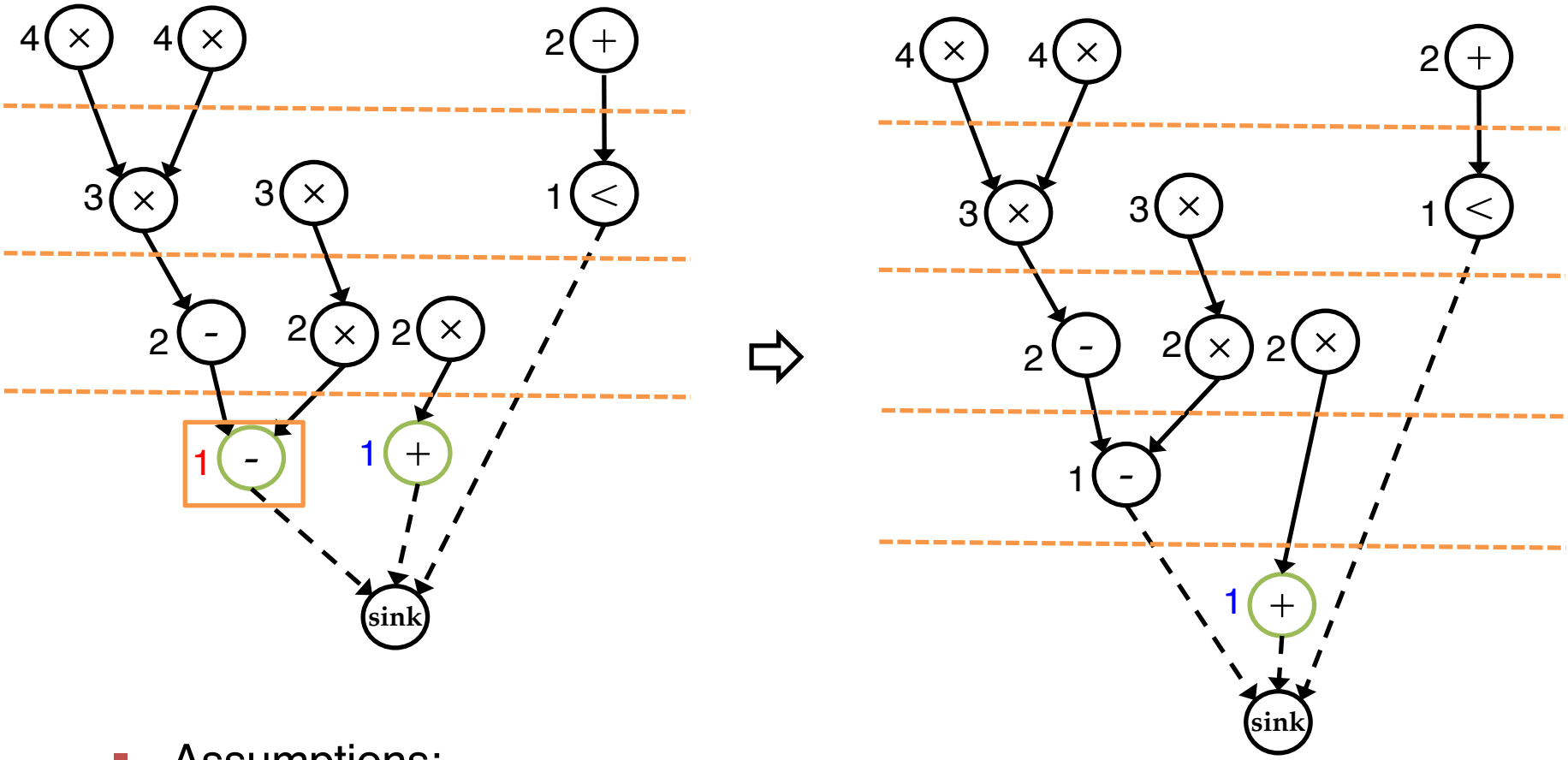
Update Ready Nodes and Repeat for Each Step



- Assumptions:
  - All operations have unit delay
  - 2 MULTs, 1 AddSub, and 1 CMP available

Repeat Until All Nodes Scheduled

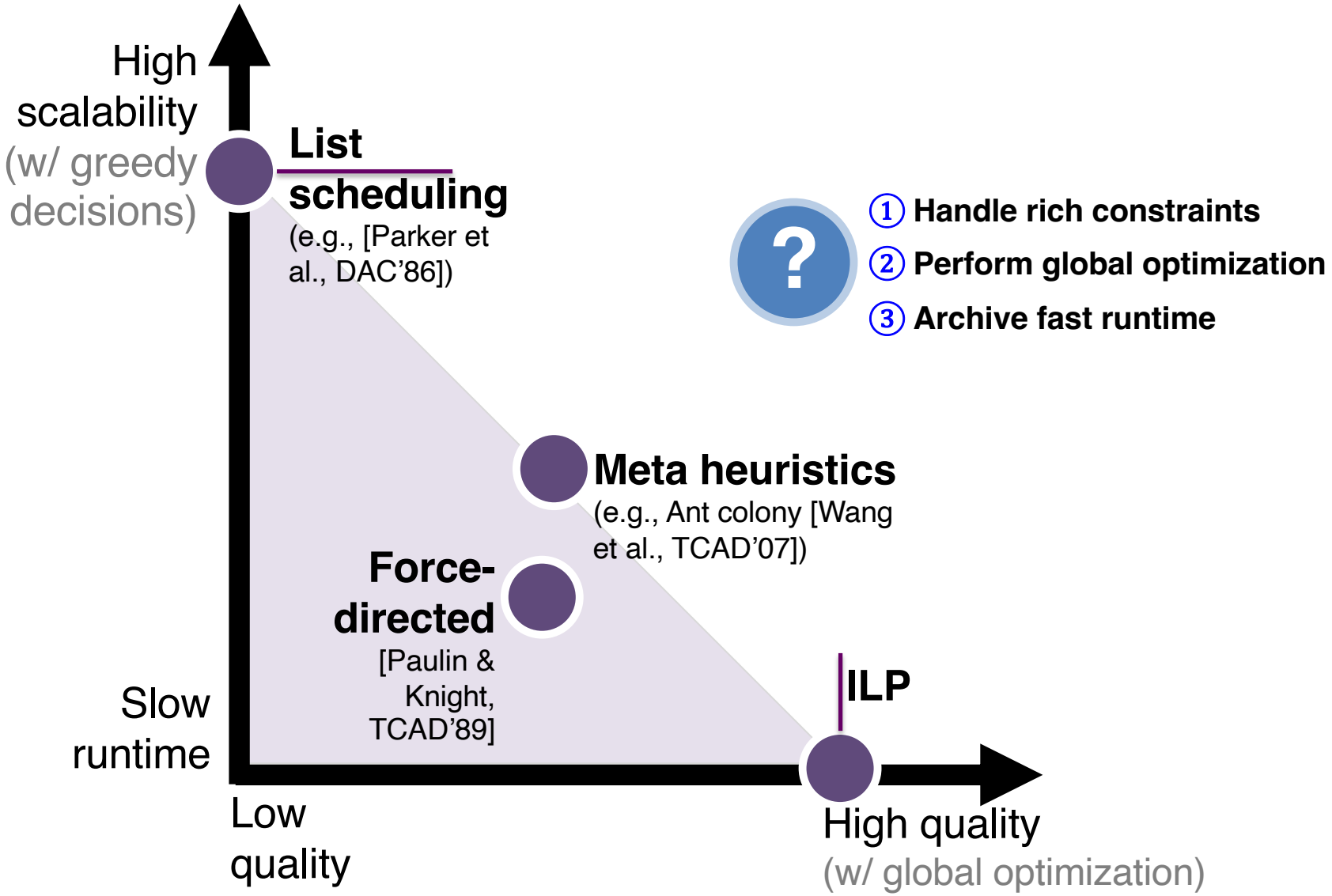


- Assumptions:
  - All operations have unit delay
  - 2 MULTs, 1 AddSub, and 1 CMP available

A Special Case

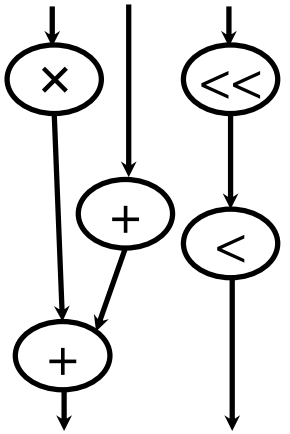
- With the following (very) restrictive conditions:
  - All operations have unit delay (i.e., single cycle)
  - All operations (and resources) are of the same type
  - Graph is a forest
- List scheduling with static height-based priorities guarantees optimality
- This is known as Hu's algorithm
  - T. C. Hu, Parallel sequencing and assembly line problems. Operations Research, 9(6), 841-848, 1961
  - Guarantees

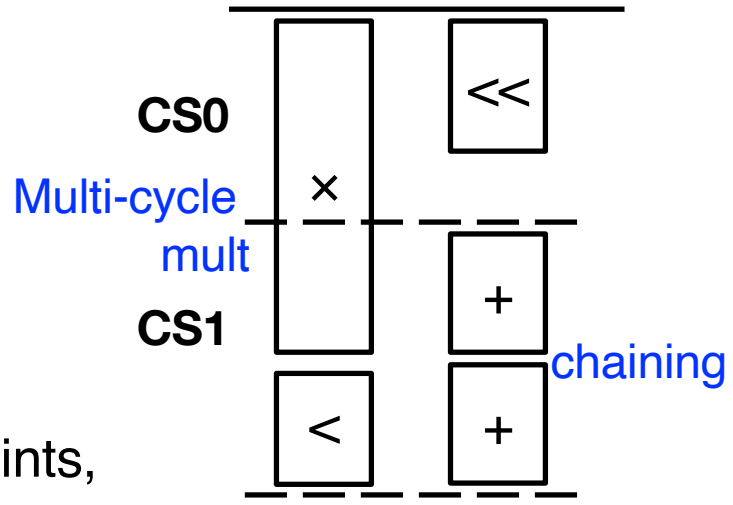
HLS Scheduling: Tension between Scalability and Quality



More Realistic Scheduling Problems

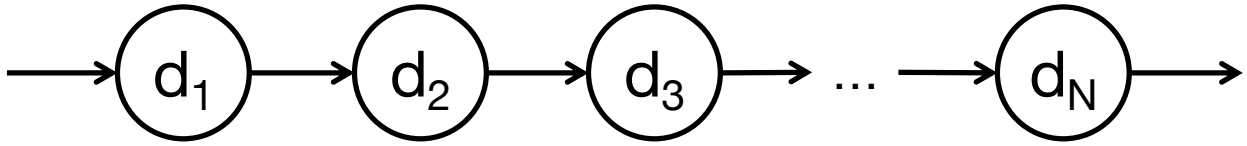
- Operation chaining
  - More compact schedule
- Multi-cycle operations
  - Nonpipelined or pipelined
  - Higher frequency
- Mutually exclusive operations
  - Scheduled in the same step, but with mutually exclusive execution conditions
  - Higher resource utilization
- Other timing constraints
  - Frequency constraints, latency constraints, relative time constraints





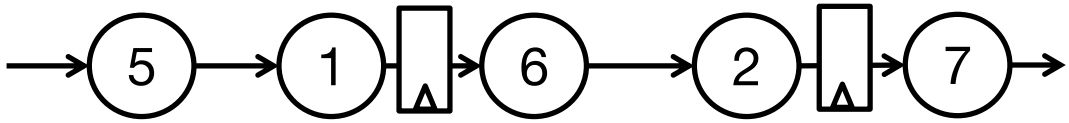
A Simple Operation Chaining Problem

Given: A chain of n operations. Without any registers, the cycle time equals the total combinational delay, which is D = sum(d_i).



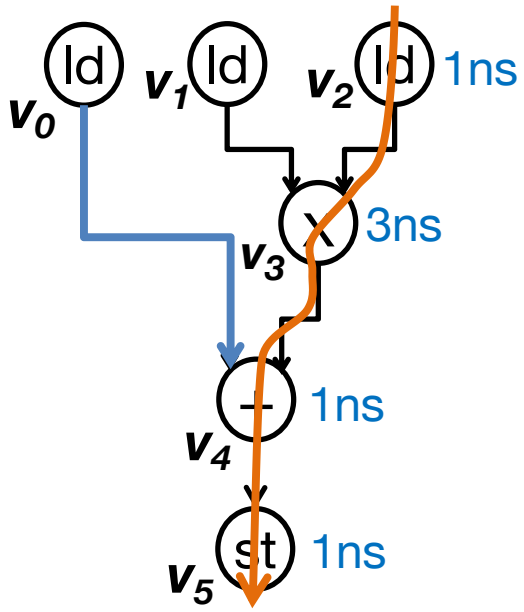
Question: How to place TWO registers on the chain to achieve the minimum cycle time?

Example:



SDC-Based Scheduling

- SDC = System of difference constraints



- Target cycle time: 5ns
- Delay estimates
  - Mul (x): 3ns
  - Add (+): 1ns
  - Load/Store (ld/st): 1ns

s_i : schedule variable for operation *i*

- Dependence constraints
  - $<v_0 , v_4> : s_0 - s_4 \le 0$
  - $<v_1 , v_3> : s_1 - s_3 \le 0$
  - $<v_2 , v_3> : s_2 - s_3 \le 0$
  - $<v_3 , v_4> : s_3 - s_4 \le 0$
  - $<v_4 , v_5> : s_4 - s_5 \le 0$

Operation chaining is naturally supported

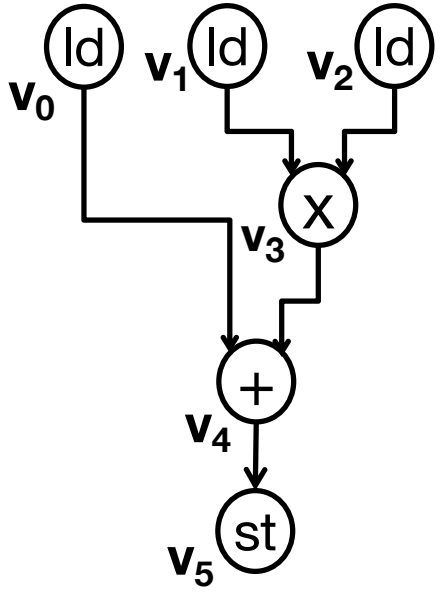
- Cycle time constraints
  - $v_1 \rightarrow v_5 : s_1 - s_5 \le -1$
  - $v_2 \rightarrow v_5 : s_2 - s_5 \le -1$

Timing constraints

To meet the cycle time, v_2 and v_5 should have a minimum separation of one cycle

[J. Cong & Z. Zhang, DAC'2006] [Z. Zhang & B. Liu, ICCAD'2013]

Exercise: Latency Constraint in SDC



How to enforce that operations v_3 and v_4 are not chained and at most two cycles apart?

Difference Constraints

- A **difference constraint** is a formula in the form of $x - y \leq b$ or $x - y < b$ for numeric variables x and y, and constant b

- With scheduling variables, we use **integer difference constraints** to model a variety of scheduling constraints
  - x and y must have integral values
    - Thus b only needs to be an integer => form $x - y < b$ is redundant

SDC Constraint Matrix

- The constraint matrix of SDC(X, C) is a totally unimodular matrix (TUM):
  - Every nonsingular square submatrix has a determinant of -1/+1.

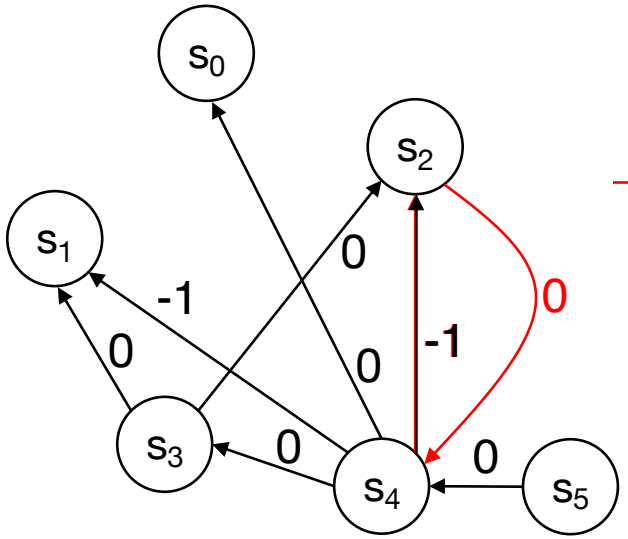
$$\begin{pmatrix} 1 & 0 & 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} s_0 \\ s_1 \\ s_2 \\ s_3 \\ s_4 \\ s_5 \end{pmatrix} \leq \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ -1 \\ -1 \end{pmatrix}$$

A x b

- Theorem (Hoffman & Kruskal, 1956): If A is totally unimodular and b is a vector of integers, every extreme point of polyhedron $\{x : Ax \leq b\}$ is integral.
  - **Solving linear programming (LP) relaxation leads to integral solutions**

SDC Constraint Graph

- Difference constraints can be conveniently represented using **constraint graph**
  - Each vertex represents a variable, and each weighted edge corresponds to a different constraint
  - Detect infeasibility by the presence of negative cycle (by solving single-source shortest path)



$s_2 - s_4 \leq -1$

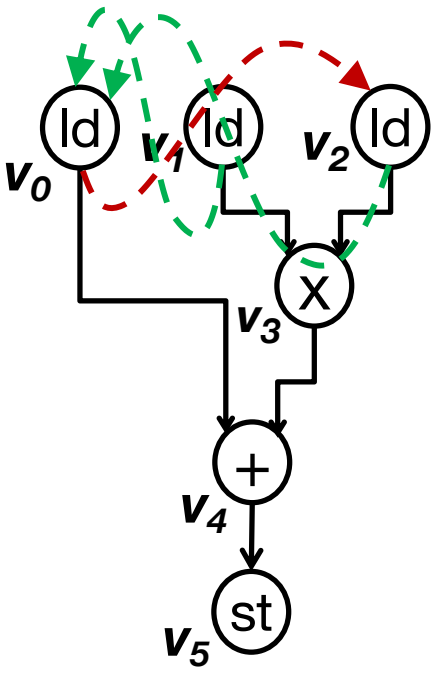
$s_4 - s_2 \leq 0$

$0 \leq -1$

$s_0 - s_4 \leq 0$
$s_1 - s_3 \leq 0$
$s_2 - s_3 \leq 0$
$s_3 - s_4 \leq 0$
$s_4 - s_5 \leq 0$
$s_2 - s_4 \leq -1$
$s_1 - s_4 \leq -1$
$s_4 - s_2 \leq 0$

Handling Resource Constraints (NP-Hard in General)

- Resource constraints cannot be represented exactly in integer difference form*



- Resource constraint
  - Two read ports

- Resource constraints
  - ➔ Heuristic partial orderings

$v_0 \rightarrow v_2 : s_0 - s_2 \leq -1$ 3 cycle latency

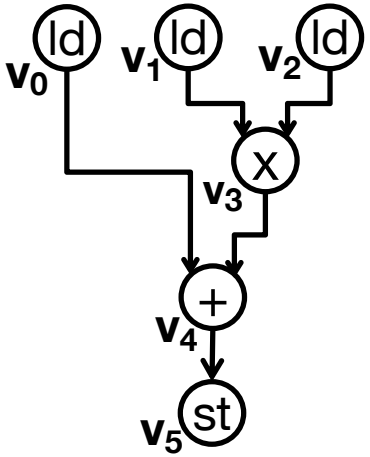
OR

$v_1 \rightarrow v_0 : s_1 - s_0 \leq -1$
$v_2 \rightarrow v_0 : s_2 - s_0 \leq -1$
2 cycle latency

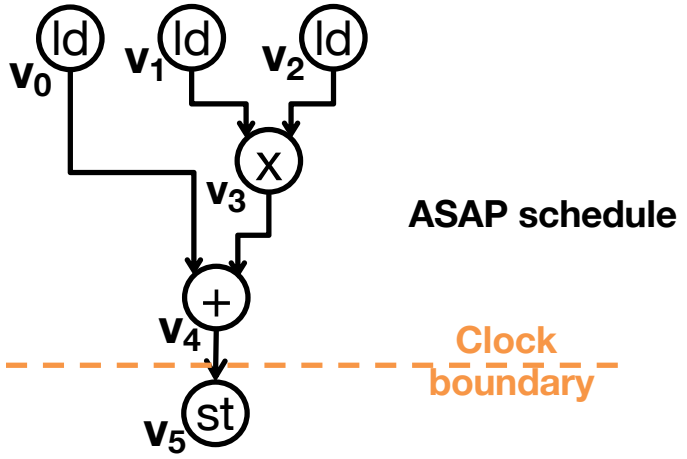
*A more recent SDC scheduling paper combined SAT and SDC to solve RCS exactly [Dai et al. FPGA'2018]

Linear Objectives

- ASAP: min $\sum_{i \in V} s_i$
- ALAP: max $\sum_{i \in V} s_i$
- Minimum latency: min $\max_{i \in V} \{s_i\}$
- Minimum average case latency (control-intensive design)
- Many other …

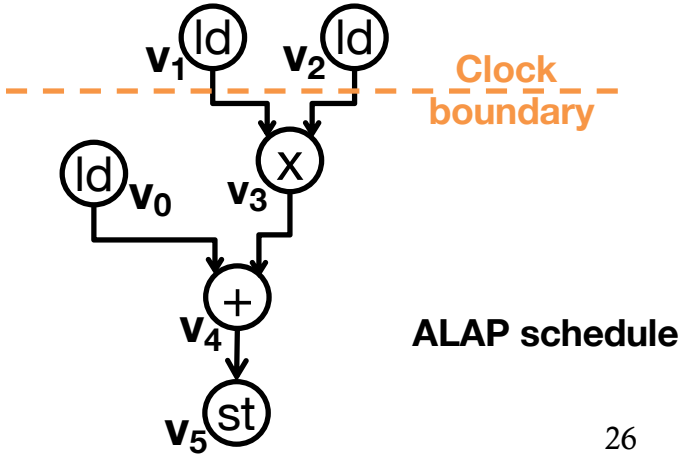


min $s_0 + \ldots + s_5$



ASAP schedule

max $s_0 + \ldots + s_5$

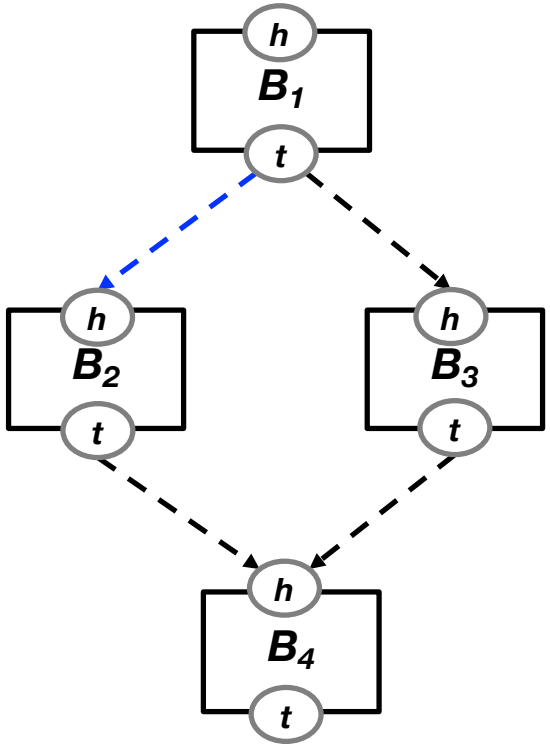


ALAP schedule

- Target cycle time: 5ns
- Delay estimates
  - Mul (x): 3ns
  - Add (+): 1ns
  - Load/Store (ld/st): 1ns

Control Flow Graphs

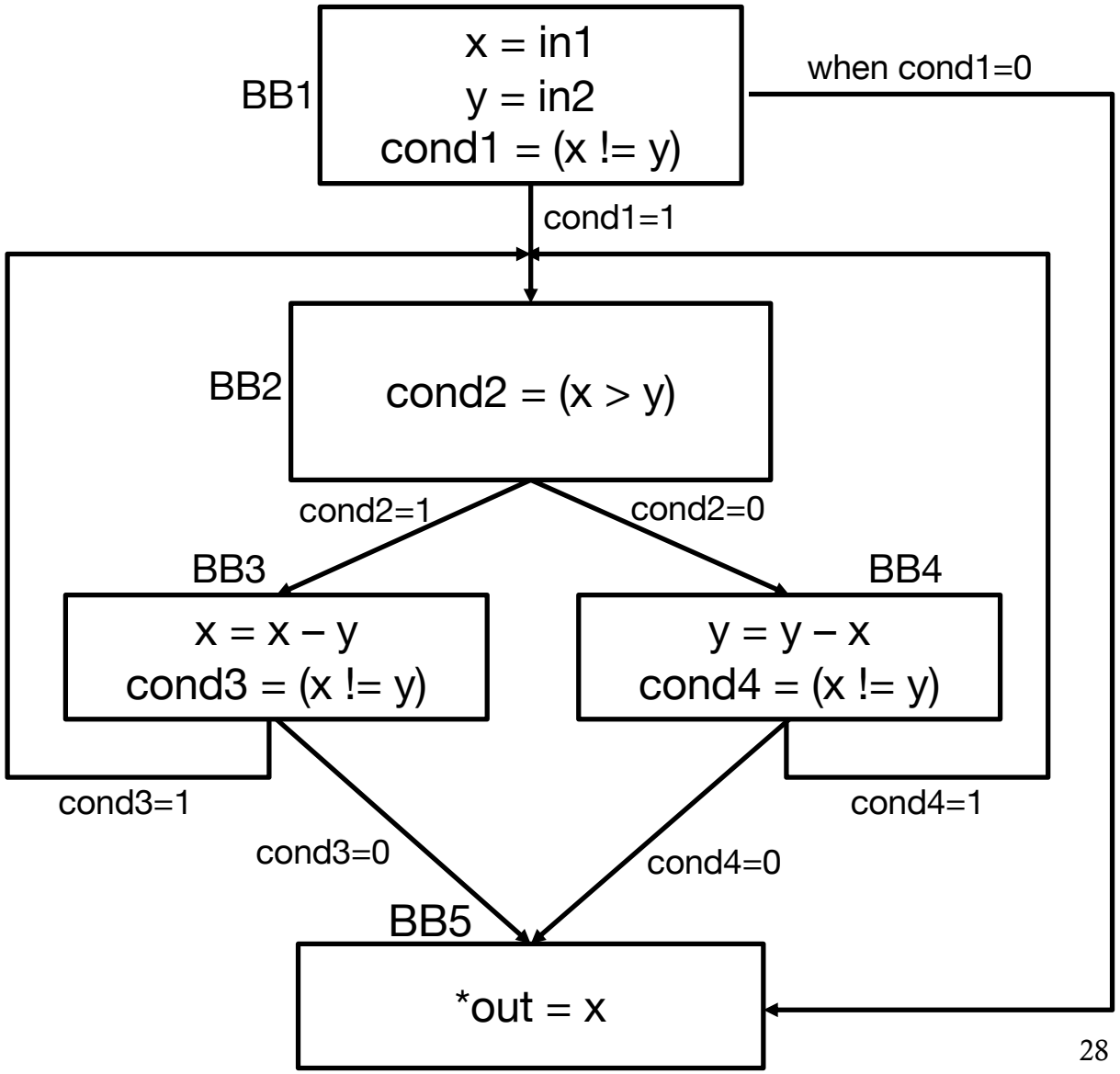
- Control dependencies can also be honored
  - If bb_2 is control dependent on bb_1 , the operation nodes of bb_2 are not allowed to be scheduled before those of bb_1
  - Polarize each basic block bb_i with two scheduling variables (head and tail)
    - $\forall v \in bb_i$, $s_h(bb_i) - s_h(v) \leq 0$
    - $\forall v \in bb_i$, $s_t(v) - s_t(bb_i) \leq 0$
  - If $e_c(bb_i, bb_j) \in E_c$ and e_c is not a back edge
    - $s_t(bb_i) - s_h(bb_j) \leq 0$



$s_t(B_1) - s_h(B_2) \leq 0$

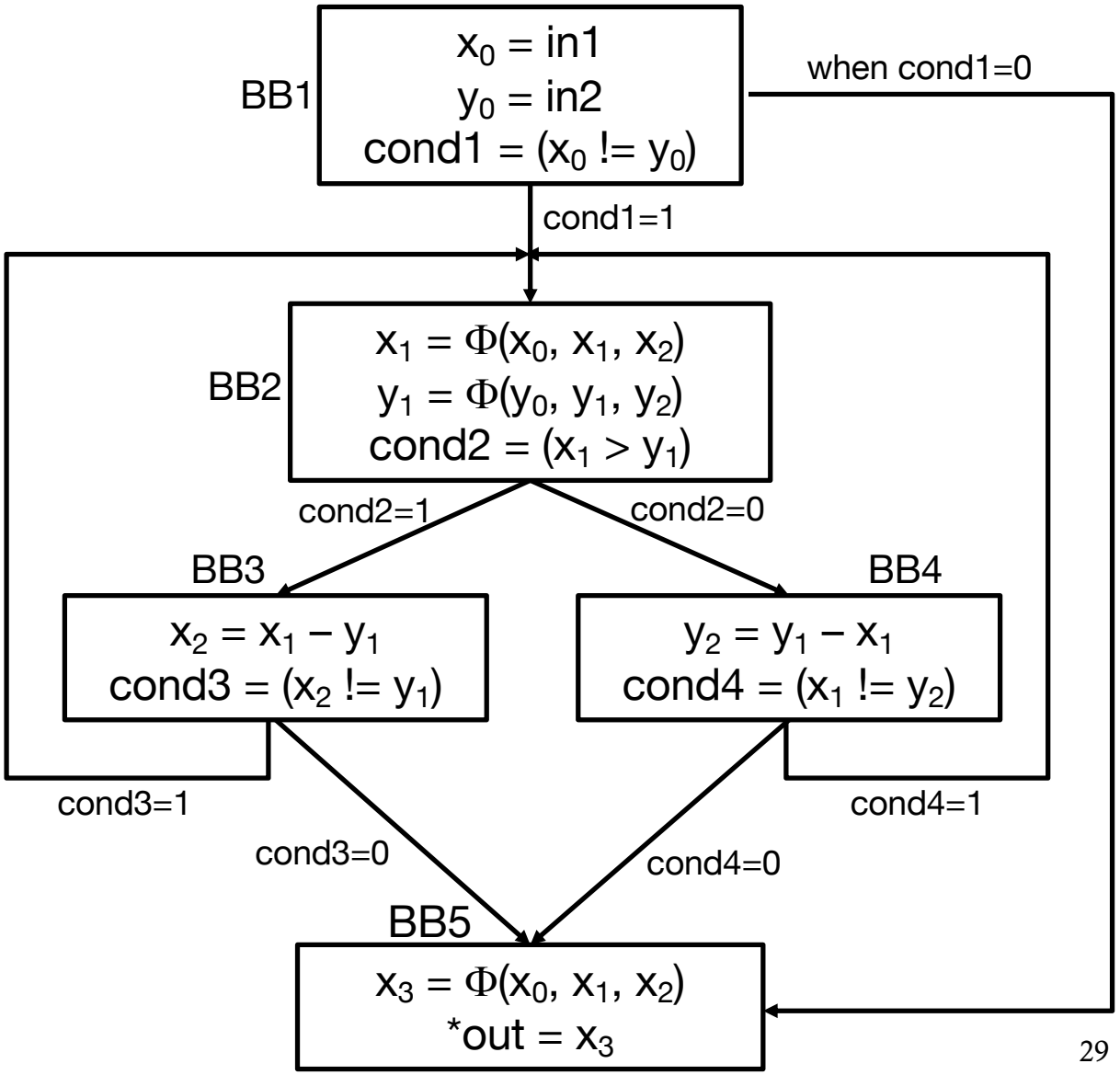
Example: Greatest Common Divisor (GCD)

```
x = in1;
y = in2;
while (x != y) {
    if ( x > y )
        x = x – y;
    else y = y – x;
}
*out = x;
```



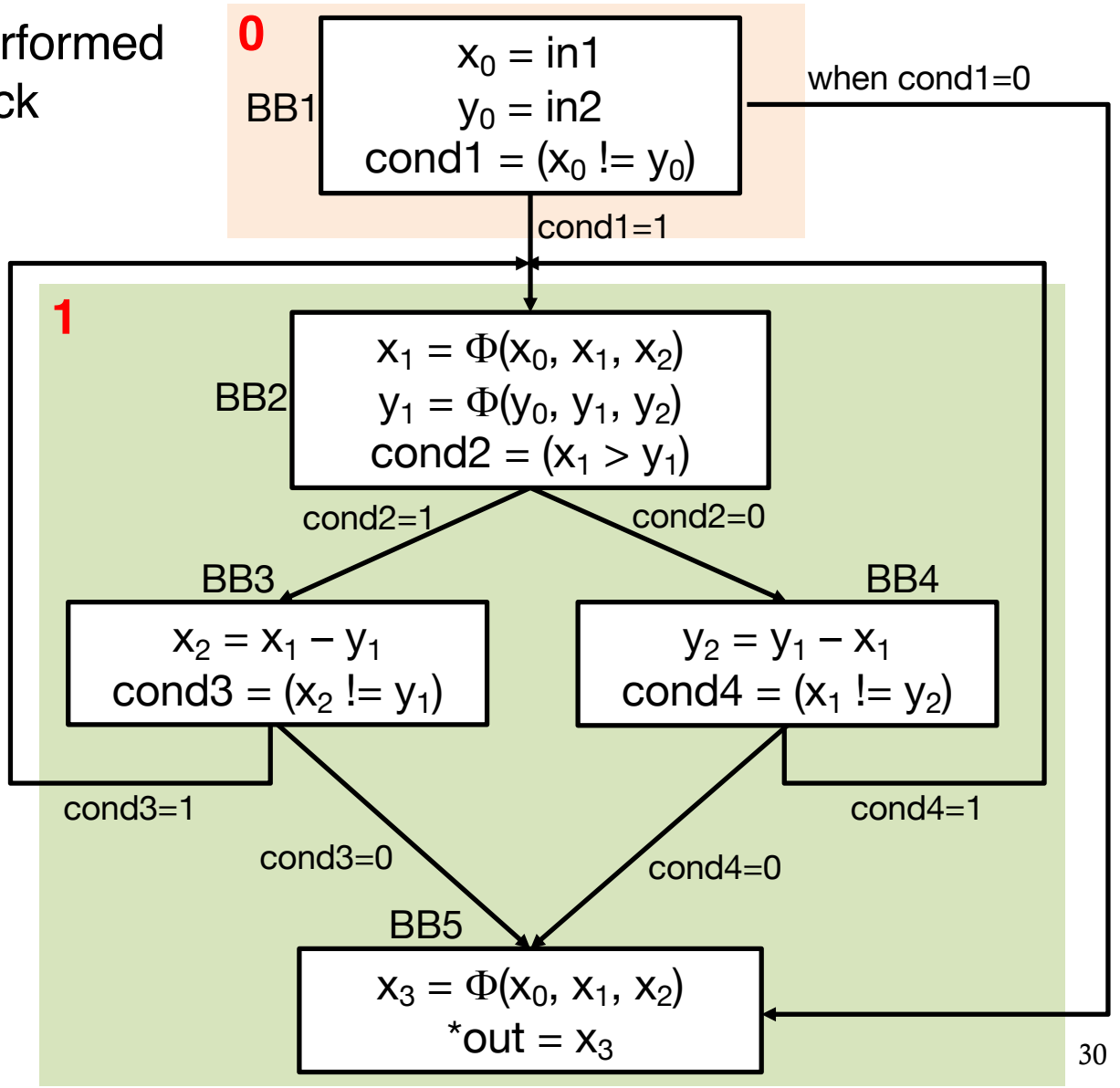
GCD in SSA form

```
x = in1;
y = in2;
while (x != y) {
    if ( x > y )
        x = x – y;
    else y = y – x;
}
*out = x;
```



Interpreting the LP Solution of SDC Scheduling

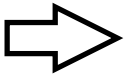
- Scheduling is performed across basic block boundaries



Operations and Predicates

0

x_0 = in1
y_0 = in2
cond1 = (x_0 != y_0)

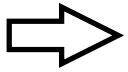


x_0 = in1
y_0 = in2
cond1 = (x_0 != y_0)

Add predicates for conditionally executed operations in each state

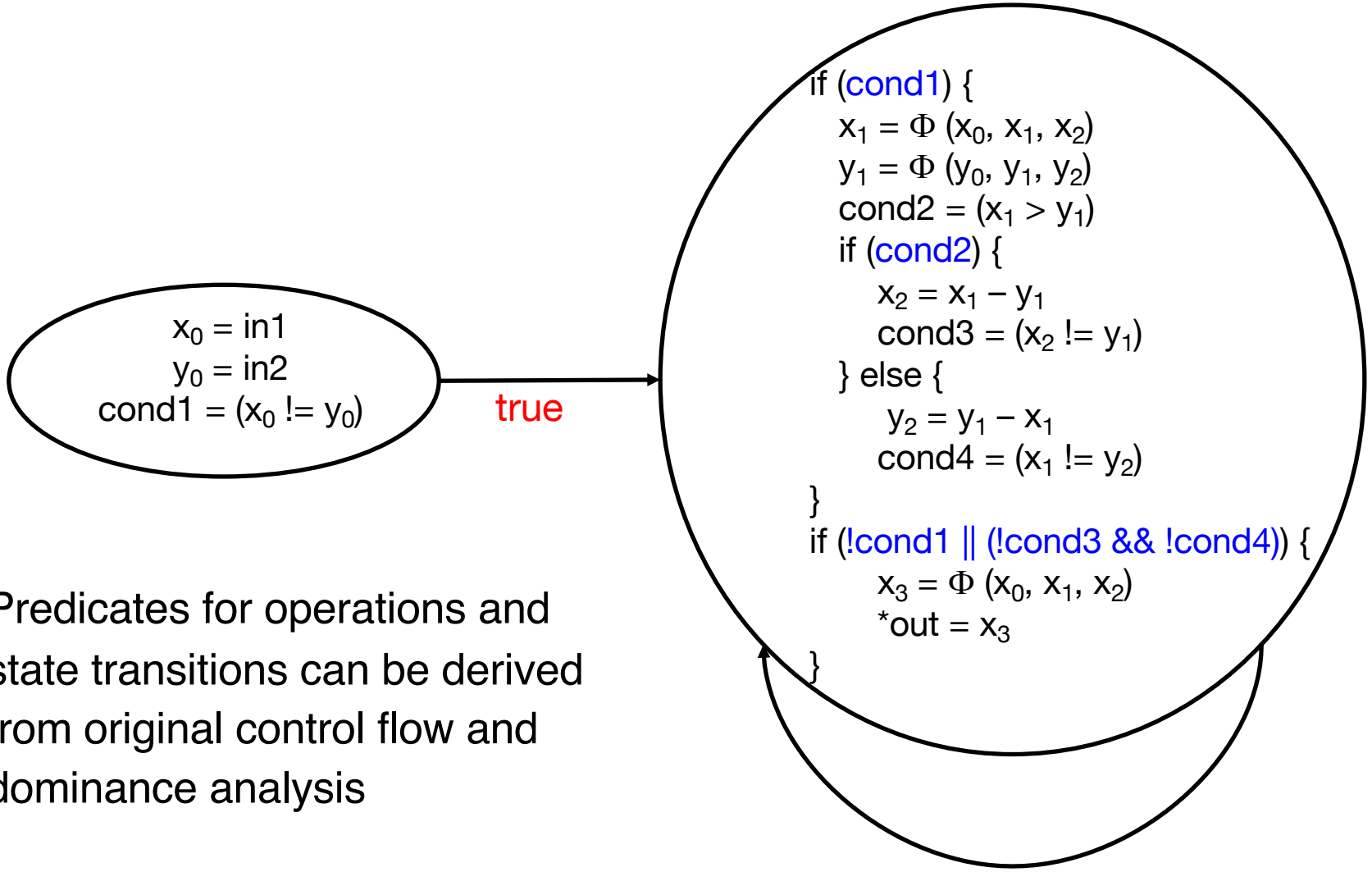
1

$x_1 = \Phi(x_0, x_1, x_2)$
$y_1 = \Phi(y_0, y_1, y_2)$
cond2 = (x_1 > y_1)
$x_2 = x_1 - y_1$
cond3 = (x_2 != y_1)
$y_2 = y_1 - x_1$
cond4 = (x_1 != y_2)
$x_3 = \Phi(x_0, x_1, x_2)$
*out = x_3



```
if (cond1) {
    x1 = Φ (x0, x1, x2)
    y1 = Φ (y0, y1, y2)
    cond2 = (x1 > y1)
    if (cond2) {
        x2 = x1 – y1
        cond3 = (x2 != y1)
    } else {
        y2 = y1 – x1
        cond4 = (x1 != y2)
}
if (!cond1 || (!cond3 && !cond4)) {
        x3 = Φ (x0, x1, x2)
        *out = x3
}
```

Deriving State Transition Graph (STG)



Predicates for operations and state transitions can be derived from original control flow and dominance analysis

Scheduling Summary

- ILP
  - Exact, but exponential worst-case runtime
- Hu's algorithm
  - Optimal and polynomial
  - Only works in very restricted cases
- List scheduling
  - Extension to Hu's for general cases
  - Greedy (fast) but suboptimal
- SDC-based scheduling
  - A versatile heuristic based on LP formulation with different constraints
  - Amenable to global optimization

Next Lecture

- Pipelining

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