ECE 6775
High-Level Digital Design Automation
Fall 2023

More Scheduling

Cornell University

Announcements

- Lab 1 graded
  - Only 2 integer bits needed for the fixed-point design
- Lab 2 due tomorrow
- Lab 3 will be released soon
- Virtual lecture next Tuesday

Agenda

- ILP for time-constrained scheduling

- Heuristic algorithms for constrained scheduling
  - List scheduling
  - SDC-based scheduling

Exercise: Formulating ILP

- Minimize the number of classrooms that the school must allocate for the following courses
- Steps to formulate the ILP
  1. Create variables
  2. Each course to be scheduled to exactly one of the preferred slots
  3. Determine the number of rooms required (by creating new derived variables)
  4. Set up the objective function

Course	Preferred Slots
A	(1) (2)
B	(1) (3)
C	(2) (3)
D	(2)

(1) 8:00 – 10:00am
(2) 10:00am – 12:00pm
(3) 12:00 – 2:00pm

① $x_{i,s}$: course i uses slot s

② $x_{A,1} + x_{A,2} = 1$
$x_{B,1} + x_{B,3} = 1$
$x_{C,2} + x_{C,3} = 1$
$x_{D,2} = 1$

③ $r_1 = x_{A,1} + x_{B,1}$
$r_2 = x_{A,2} + x_{C,2} + x_{D,2}$
$r_3 = x_{B,3} + x_{C,3}$

④ Objective: min $\boxed{\max \{r1, r2, r3\}}$

$$\Big\downarrow \text{Linearize}$$

min R
$R \geq r_1$, $R \geq r_2$, $R \geq r_3$

Time-Constrained Scheduling (TCS)

- Dual problem of resource-constrained scheduling
  - Overall latency is given as a constraint (deadline)
  - Minimize the total cost in terms of area (or resource usage), power, etc.
- NP-hard problem
  - ILP formulation is exact but is not a polynomial-time solution
  - Force-directed scheduling is a well-known heuristic for TCS (see De Micheli chapter 5.4.4)

Example: ILP Formulation for TCS

- ILP for time-constrained scheduling

minimize $c^T y$

$x_{1,1} + x_{2,1} + x_{6,1} + x_{8,1} \leq y_1$

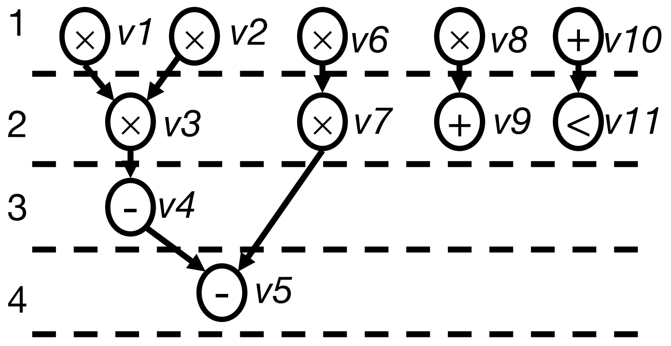
$x_{3,2} + x_{6,2} + x_{7,2} + x_{8,2} \leq y_1$

$x_{7,3} + x_{8,3} \leq y_1$

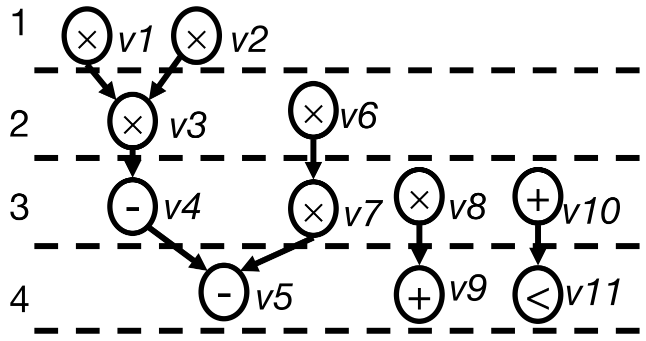
$x_{5,4} + x_{9,4} + x_{11,4} \leq y_2$

…

What does the *y* vector represent?



ASAP schedule



ALAP schedule

Recap: Constrained Scheduling in HLS

- Constrained scheduling
  - General case NP-hard
  - Resource-constrained scheduling (RCS)
    - Minimize latency given constraints on area or resources
  - Time-constrained scheduling (TCS)
    - Minimize resources subject to bound on latency

- Exact methods
  - Integer linear programming (ILP)
  - Hu's algorithm for a very restricted problem

- Heuristics
  - <u>List scheduling</u>
  - Force-directed list scheduling
  - <u>SDC-based scheduling</u>

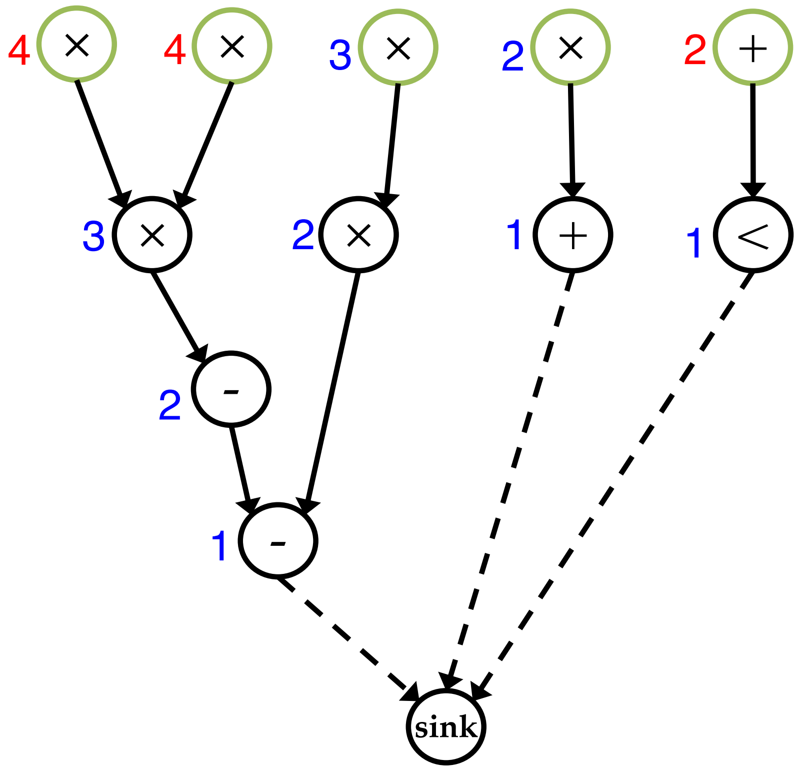
  …

List Scheduling

- A widely-used heuristic algorithm for RCS
  - Schedule one control step (cycle) at a time
  - Maintain a list of “ready” operations considering dependence
  - Assign priorities to operations; most “critical” operations (with the highest priorities) go first

- Often refers to a family of algorithms
  - Typically classified by the way priority function is calculated
    - Static priority: Priorities are calculated once before scheduling
    - Dynamic priority calculation: Priorities are updated during scheduling

Static Priority Example: Node Height

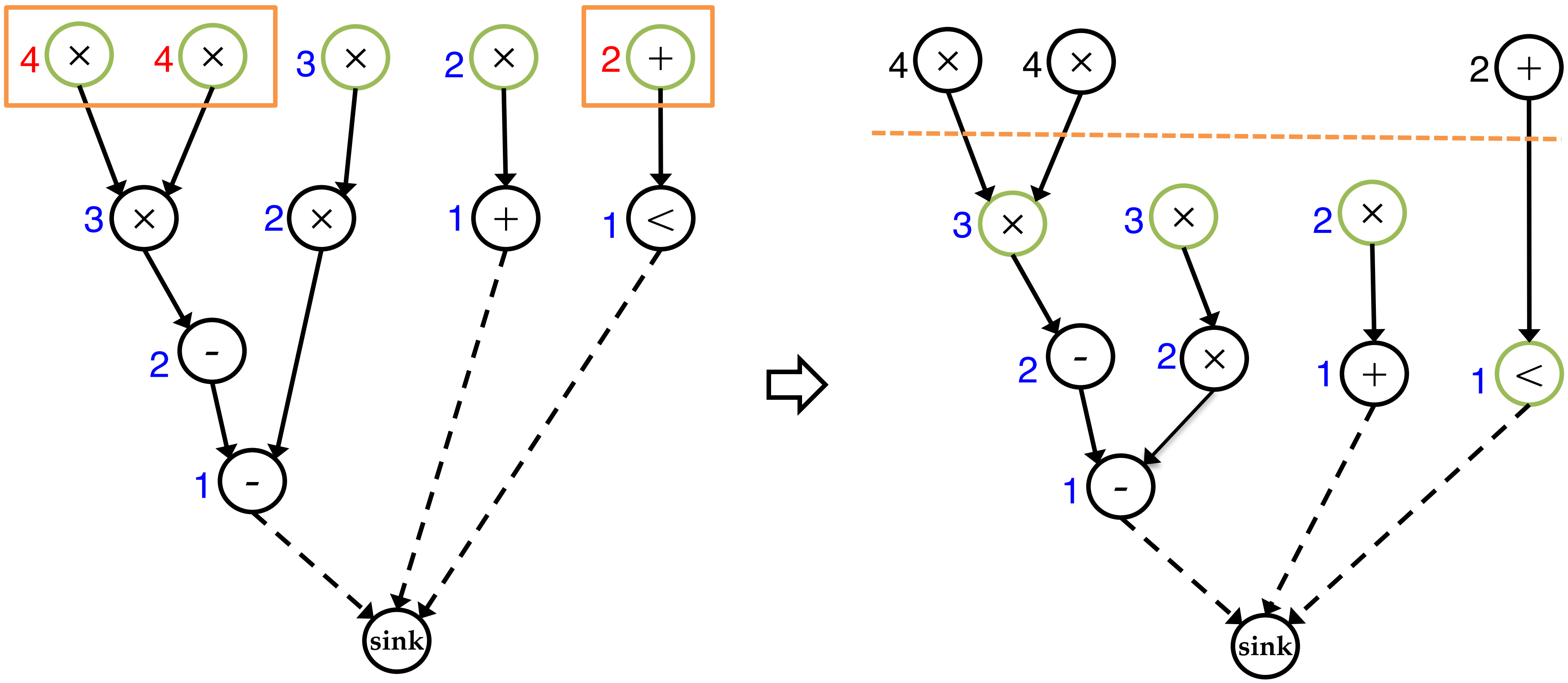


Nodes are labelled with distance to sink (height)

Ready operations are colored in green

- Assumptions:
  - All operations have unit delay
  - 2 MULTs, 1 AddSub, and 1 CMP available

Ready Nodes with Highest Priorities Picked First



- Assumptions:
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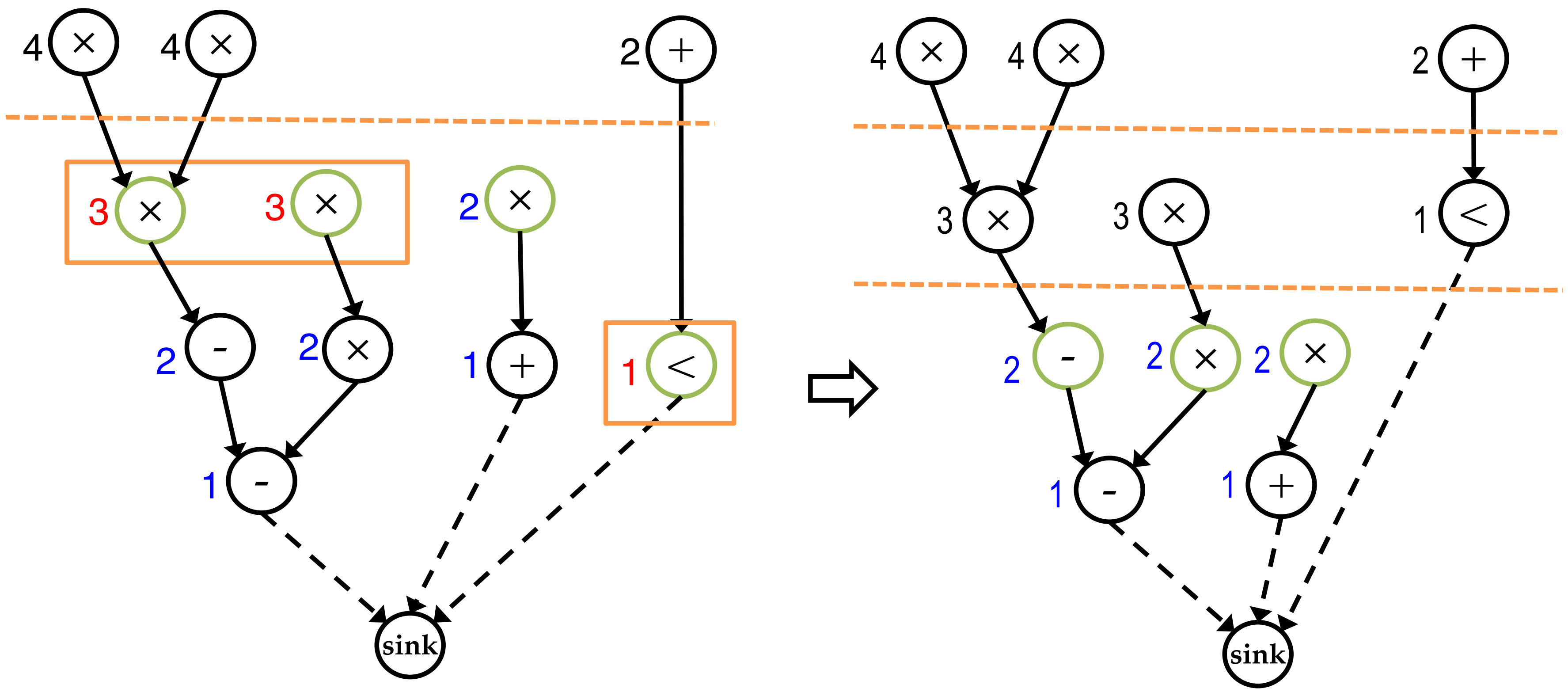
  - 2 MULTs, 1 AddSub, and 1 CMP available

Update Ready Nodes and Repeat for Each Step



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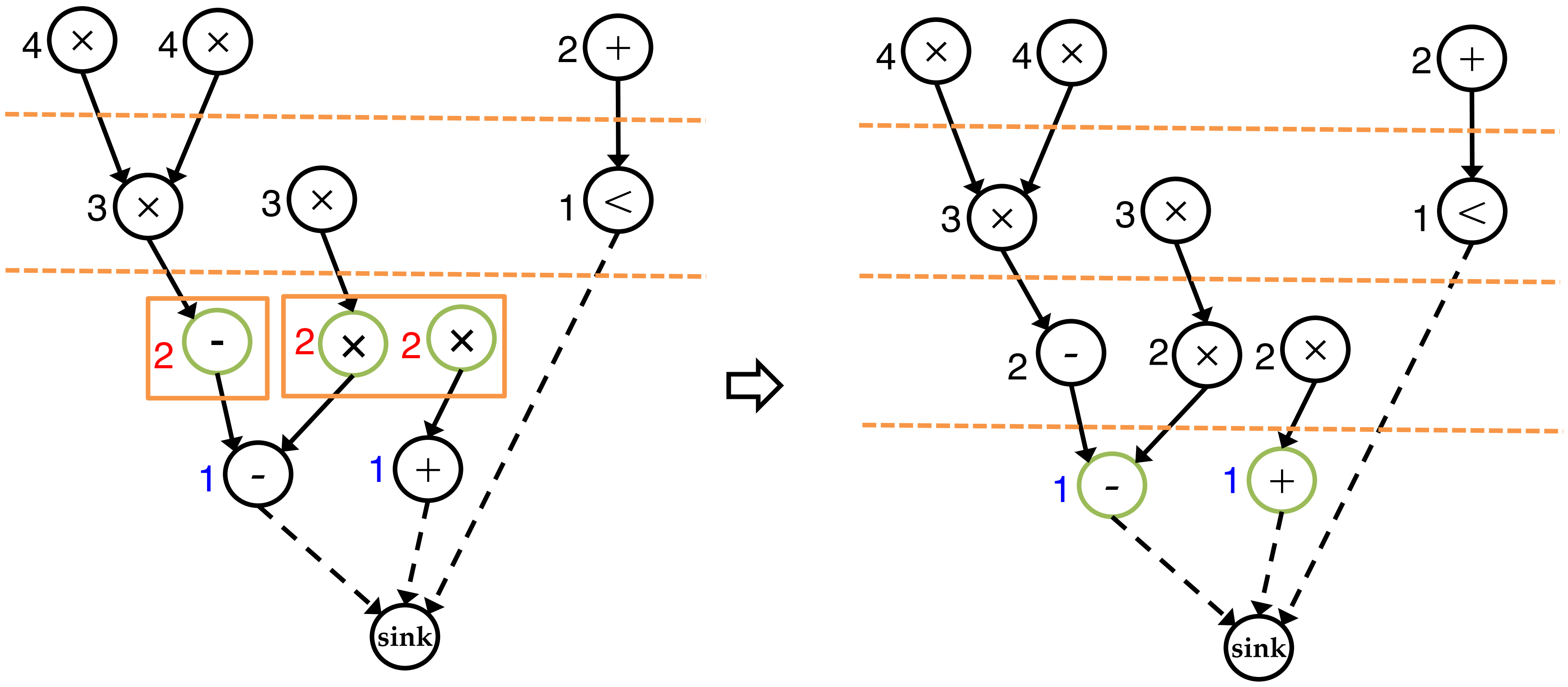
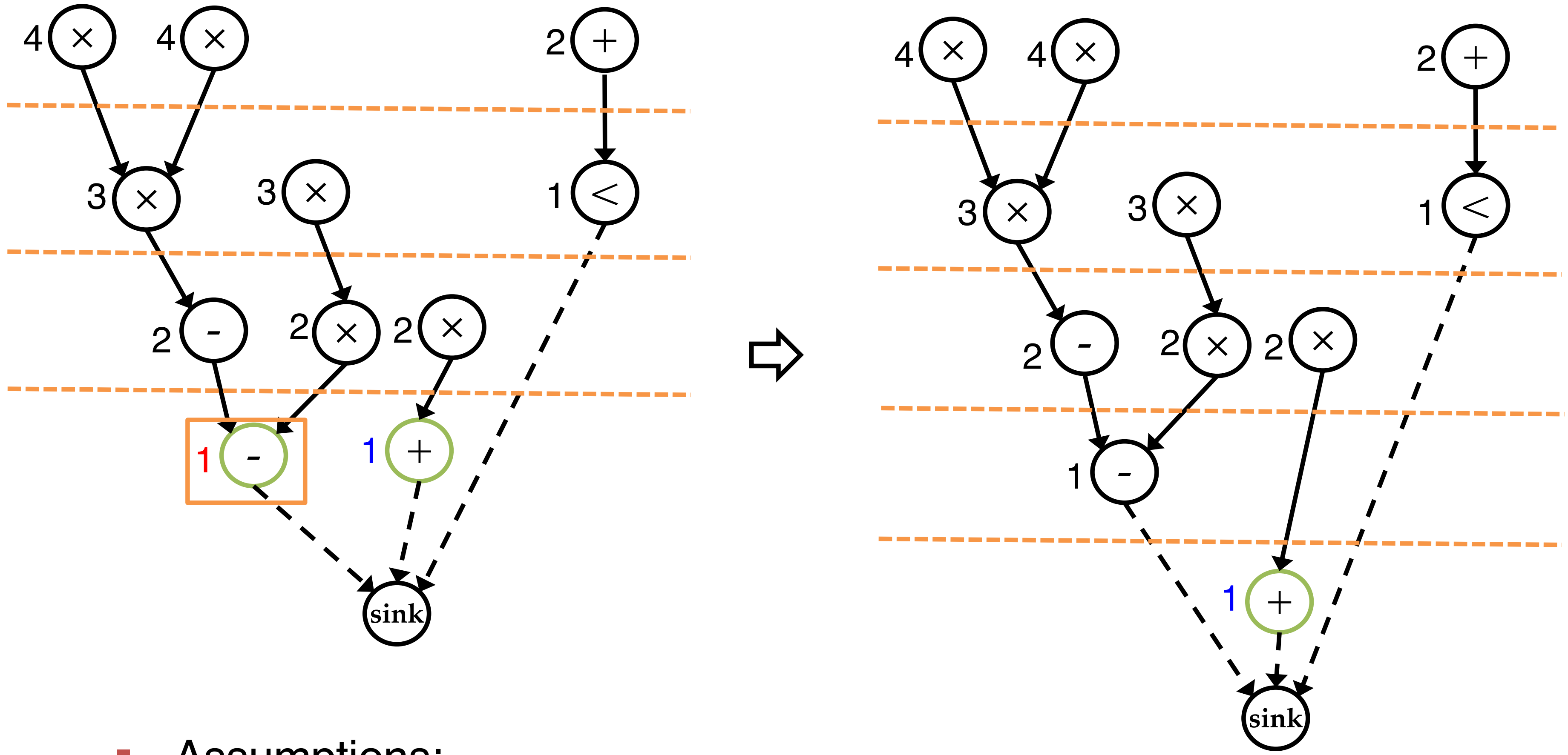
Update Ready Nodes and Repeat for Each Step



- Assumptions:
  - All operations have unit delay
  - 2 MULTs, 1 AddSub, and 1 CMP available

Repeat Until All Nodes Scheduled



- Assumptions:
  - All operations have unit delay
  - 2 MULTs, 1 AddSub, and 1 CMP available

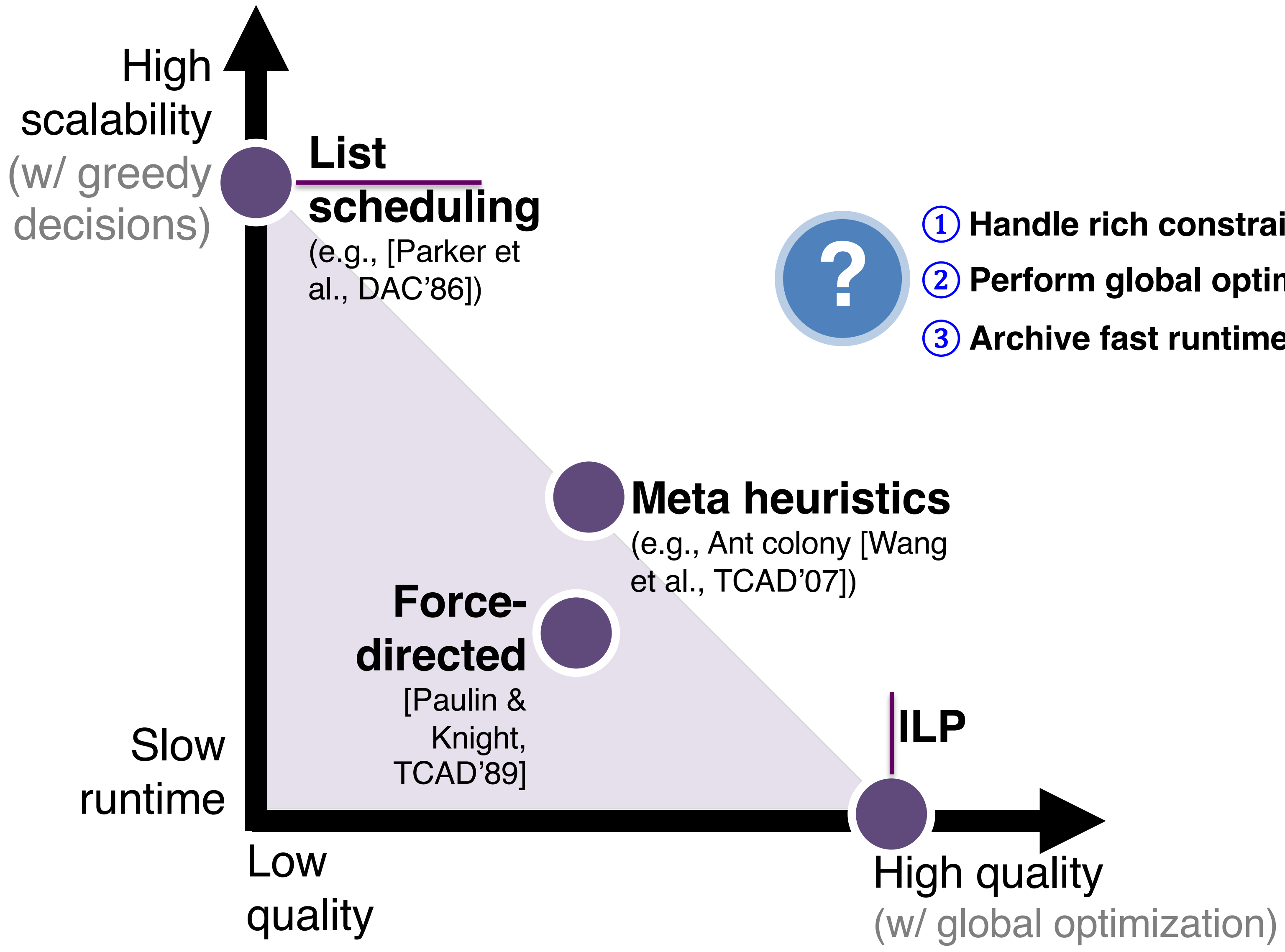
A Special Case

- With the following (very) restrictive conditions:
  - All operations have unit delay (i.e., single cycle)
  - All operations (and resources) are of the same type
  - Graph is a forest

- List scheduling with static height-based priorities guarantees optimality

- This is known as Hu’s algorithm
  - T. C. Hu, Parallel sequencing and assembly line problems. Operations Research, 9(6), 841-848, 1961
  - Guarantees

HLS Scheduling: Tension between Scalability and Quality



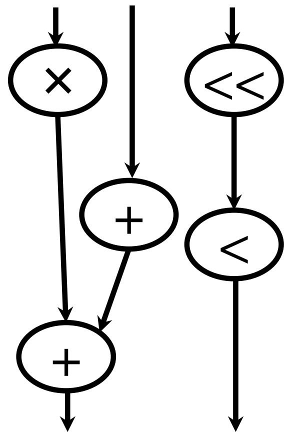
① **Handle rich constraints**

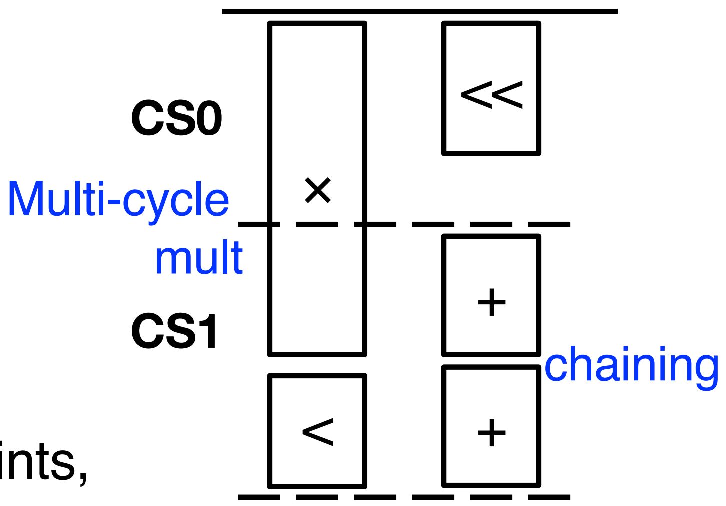
② **Perform global optimization**

③ **Archive fast runtime**

More Realistic Scheduling Problems

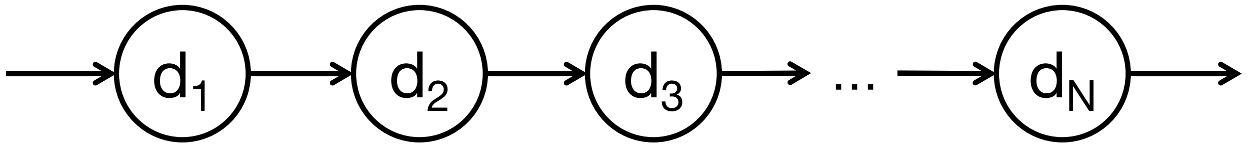
- Operation chaining
  - More compact schedule
- Multi-cycle operations
  - Nonpipelined or pipelined
  - Higher frequency
- Mutually exclusive operations
  - Scheduled in the same step, but with mutually exclusive execution conditions
  - Higher resource utilization
- Other timing constraints
  - Frequency constraints, latency constraints, relative time constraints





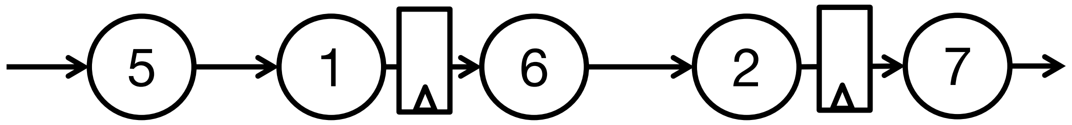
A Simple Operation Chaining Problem

Given: A chain of n operations. Without any registers, the cycle time equals the total combinational delay, which is D = sum(d_i).



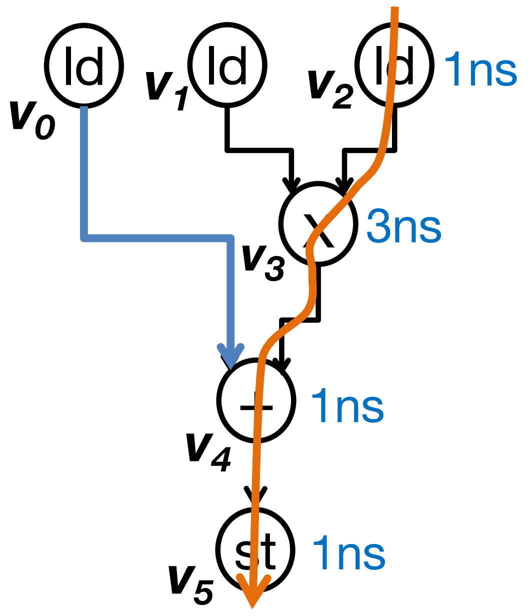
Question: How to place TWO registers on the chain to achieve the minimum cycle time?

Example:



SDC-Based Scheduling

- SDC = System of difference constraints



- Target cycle time: 5ns
- Delay estimates
  - Mul (x): 3ns
  - Add (+): 1ns
  - Load/Store (ld/st): 1ns

$\boldsymbol{s_i}$: schedule variable for operation *i*

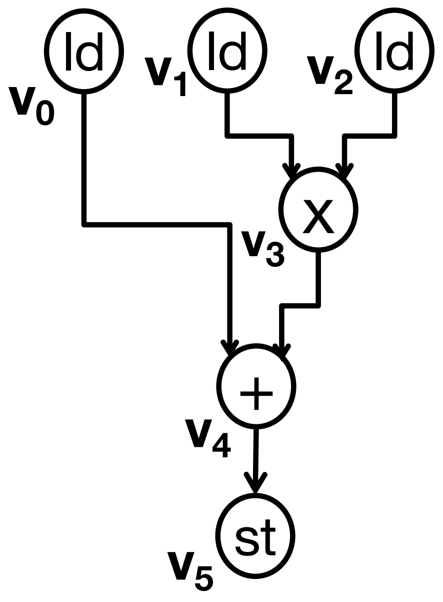
Operation chaining is naturally supported

- Dependence constraints
  - $<v_0, v_4> : s_0 - s_4 \leq 0$
  - $<v_1, v_3> : s_1 - s_3 \leq 0$
  - $<v_2, v_3> : s_2 - s_3 \leq 0$
  - $<v_3, v_4> : s_3 - s_4 \leq 0$
  - $<v_4, v_5> : s_4 - s_5 \leq 0$
- Cycle time constraints
  - $v_1 \rightarrow v_5 : s_1 - s_5 \leq -1$
  - $v_2 \rightarrow v_5 : \boxed{s_2 - s_5 \leq -1}$

Timing constraints

To meet the cycle time, v_2 and v_5 should have a minimum separation of one cycle

Exercise: Latency Constraint in SDC



How to enforce that operations $\mathbf{v_3}$ and $\mathbf{v_4}$ are not chained and at most two cycles apart?

Difference Constraints

- A **difference constraint** is a formula in the form of $x - y \leq b$ or $x - y < b$ for numeric variables x and y, and constant b

- With scheduling variables, we use **integer difference constraints** to model a variety of scheduling constraints
  - x and y must have integral values
    - Thus b only needs to be an integer => form $x - y < b$ is redundant

SDC Constraint Matrix

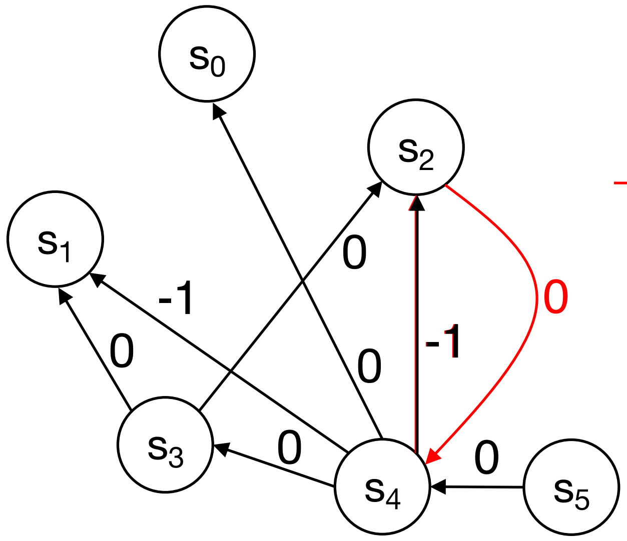
- The constraint matrix of SDC(X, C) is a totally unimodular matrix (TUM):
  - Every nonsingular square submatrix has a determinant of -1/+1.

$$\underbrace{\begin{bmatrix} 1 & 0 & 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 & 0 & -1 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} s_0 \\ s_1 \\ s_2 \\ s_3 \\ s_4 \\ s_5 \end{bmatrix}}_{x} \leq \underbrace{\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ -1 \\ -1 \end{bmatrix}}_{b}$$

- Theorem (Hoffman & Kruskal, 1956): If A is totally unimodular and b is a vector of integers, every extreme point of polyhedron $\{x : Ax \leq b\}$ is integral.
  - **Solving linear programming (LP) relaxation leads to integral solutions**

SDC Constraint Graph

- Difference constraints can be conveniently represented using **constraint graph**
  - Each vertex represents a variable, and each weighted edge corresponds to a different constraint
  - Detect infeasibility by the presence of negative cycle (by solving single-source shortest path)



$$\begin{array}{r} s_2 - s_4 \leq -1 \\ s_4 - s_2 \leq 0 \\ \hline 0 \leq -1 \end{array}$$

$s_0 - s_4 \leq 0$

$s_1 - s_3 \leq 0$

$s_2 - s_3 \leq 0$

$s_3 - s_4 \leq 0$

$s_4 - s_5 \leq 0$

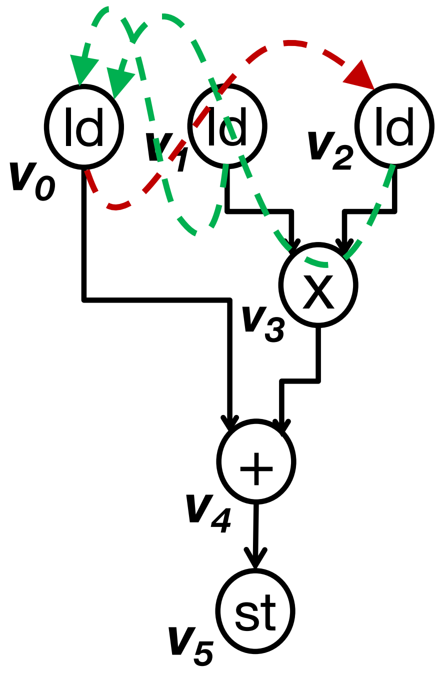
$s_2 - s_4 \leq -1$

$s_1 - s_4 \leq -1$

$s_4 - s_2 \leq 0$

Handling Resource Constraints (NP-Hard in General)

- Resource constraints cannot be represented exactly in integer difference form



- Resource constraint
  - Two read ports

- Resource constraints
  ➔ Heuristic partial orderings

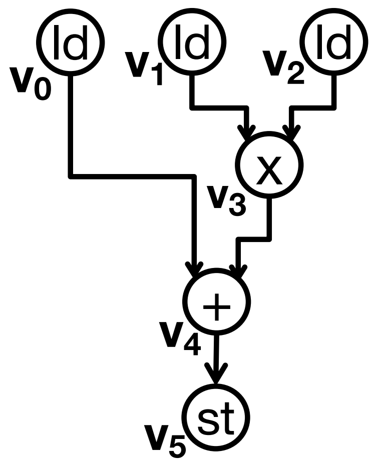
$v_0 \rightarrow v_2 : s_0 - s_2 \leq -1$ 3 cycle latency

OR

$v_1 \rightarrow v_0 : s_1 - s_0 \leq -1$
$v_2 \rightarrow v_0 : s_2 - s_0 \leq -1$ 2 cycle latency

Linear Objectives

- ASAP: min $\sum_{i \in V} s_i$
- ALAP: max $\sum_{i \in V} s_i$
- Minimum latency: min $\max_{i \in V} \{s_i\}$
- Minimum average case latency (control-intensive design)
- Many other …



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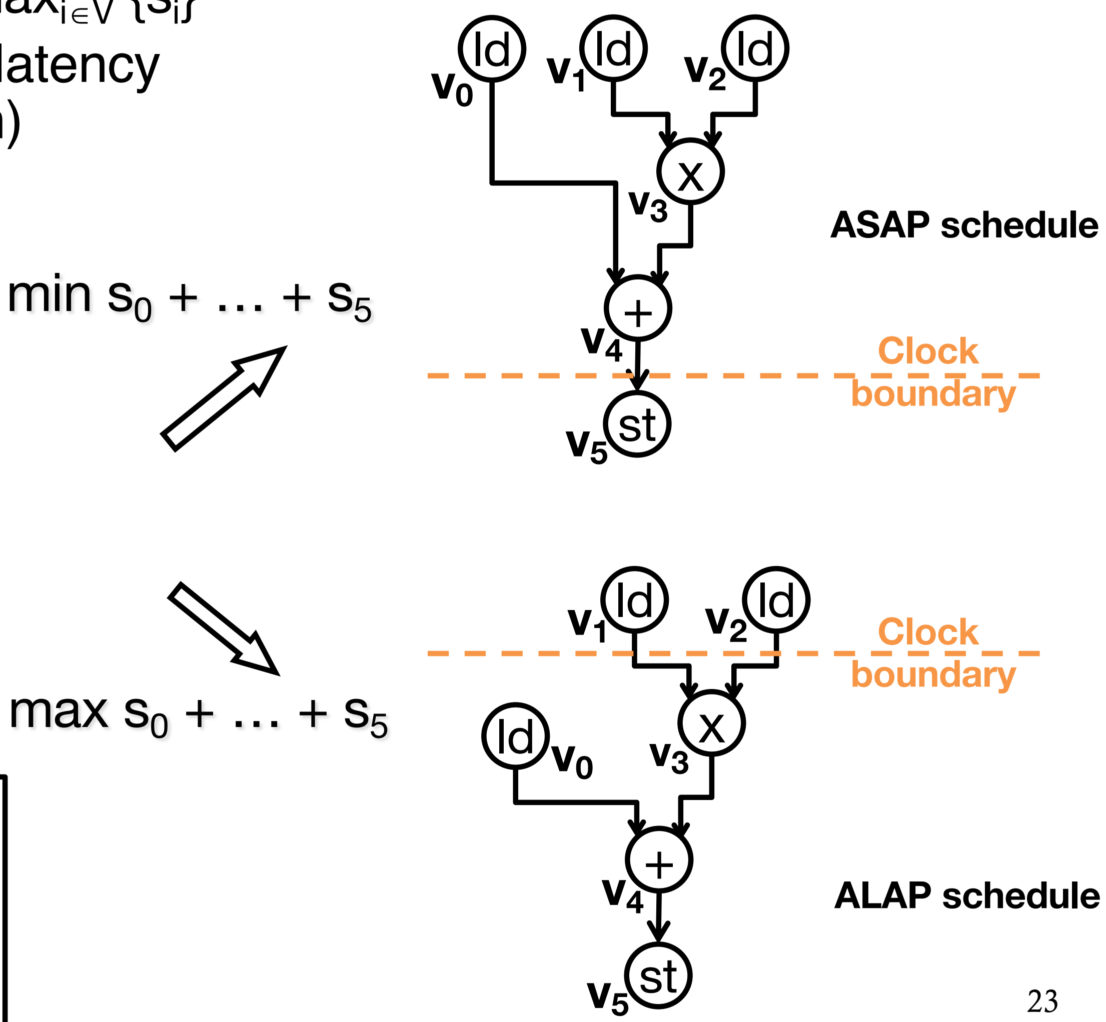
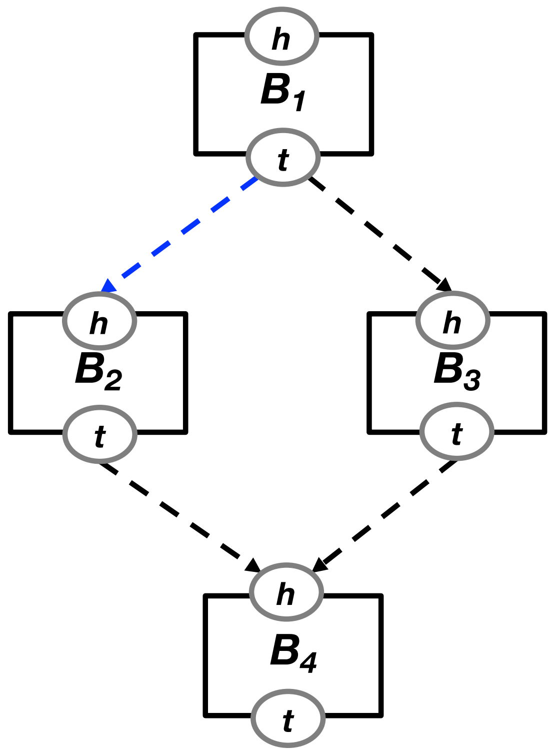


Control Flow Graphs

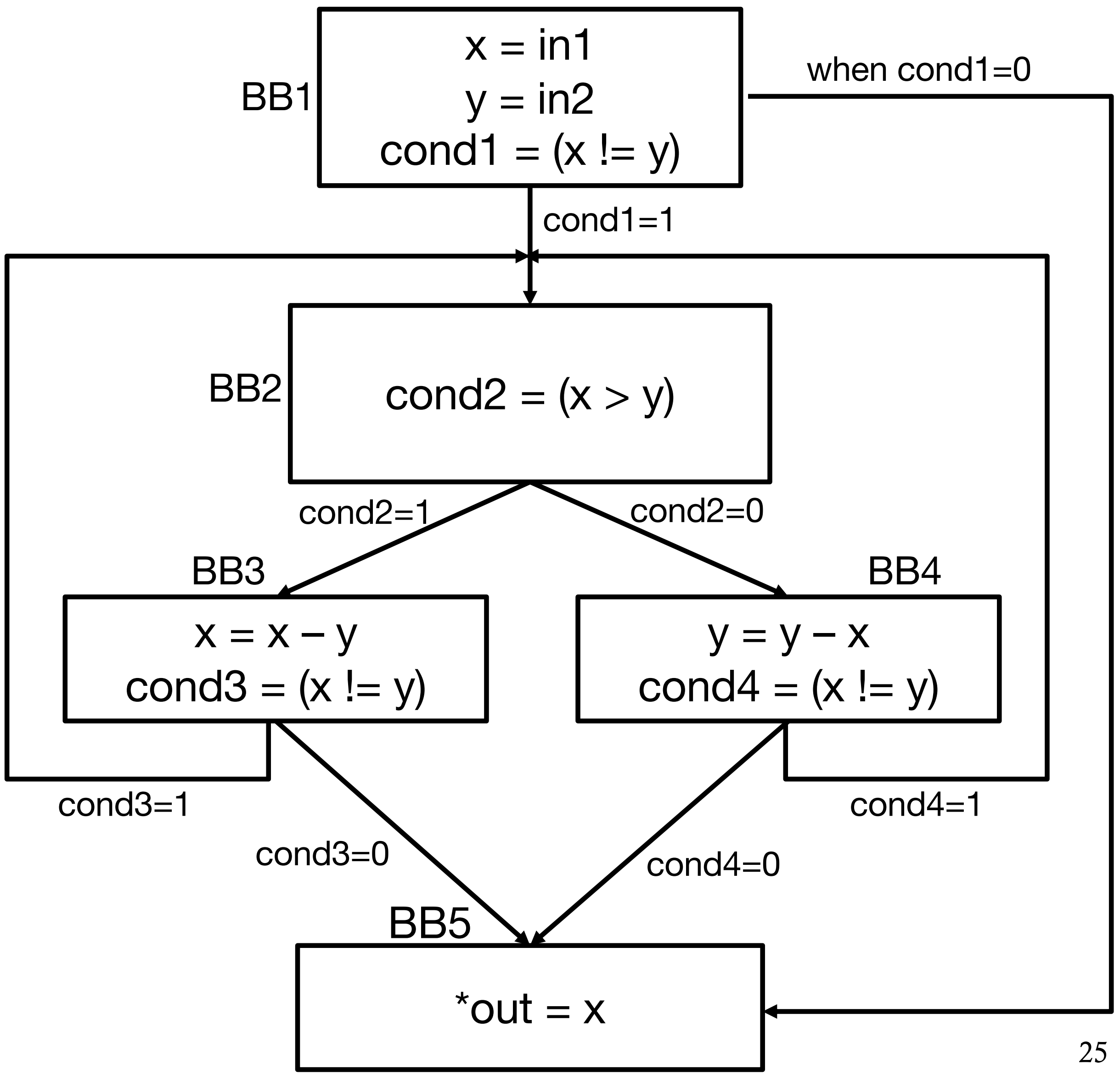
- Control dependencies can also be honored
  - If bb_2 is control dependent on bb_1, the operation nodes of bb_2 are not allowed to be scheduled before those of bb_1
  - Polarize each basic block bb_i with two scheduling variables (head and tail)
    - $\forall v \in bb_i$, $s_h(bb_i) - s_h(v) \leq 0$
    - $\forall v \in bb_i$, $s_t(v) - s_t(bb_i) \leq 0$
  - If $e_c(bb_i, bb_j) \in E_c$ and e_c is not a back edge
    - $s_t(bb_i) - s_h(bb_j) \leq 0$



$s_t(B_1) - s_h(B_2) \leq 0$

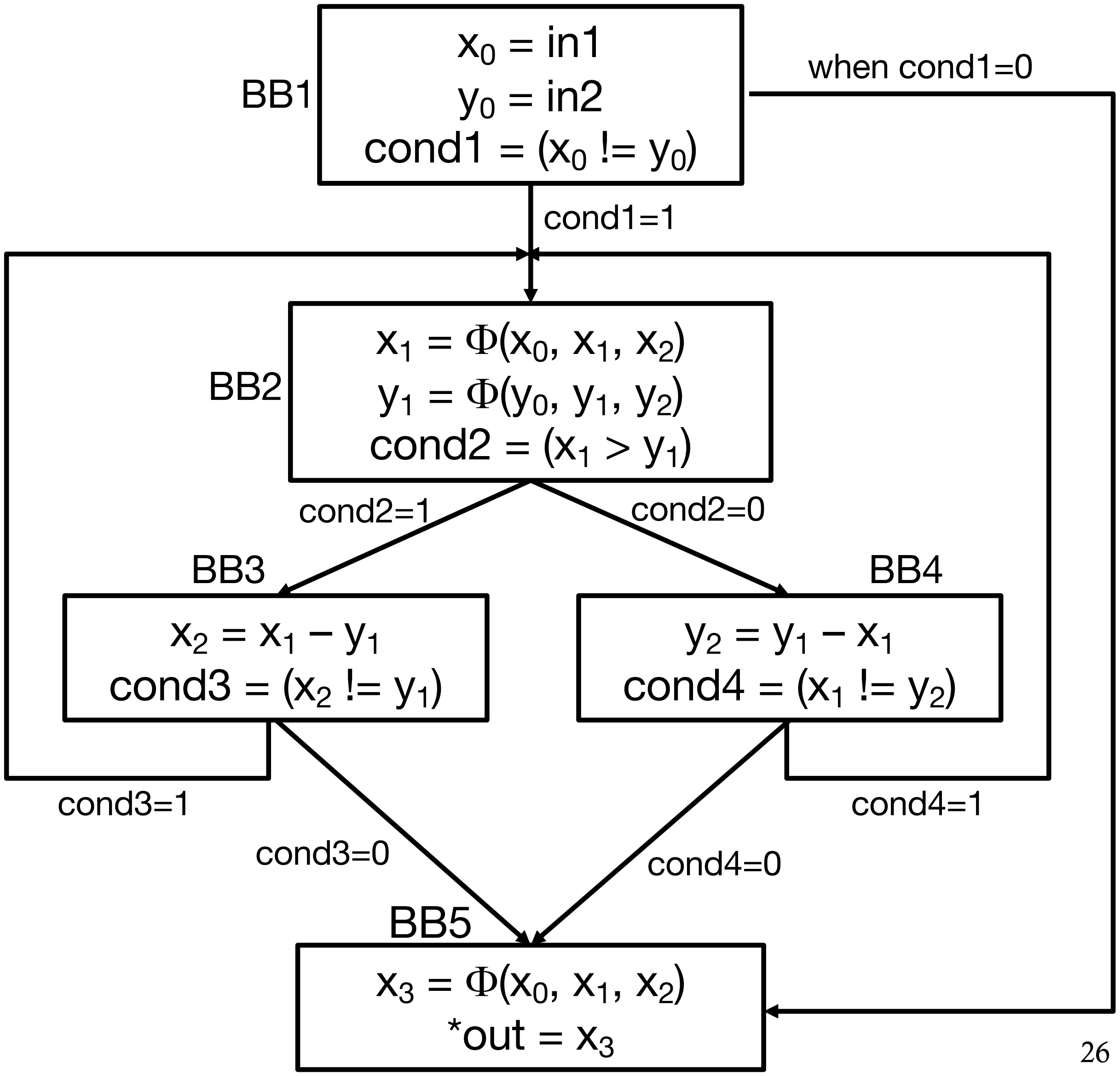
Example: Greatest Common Divisor (GCD)

```
x = in1;
y = in2;
while (x != y) {
    if ( x > y )
        x = x – y;
    else y = y – x;
}
*out = x;
```



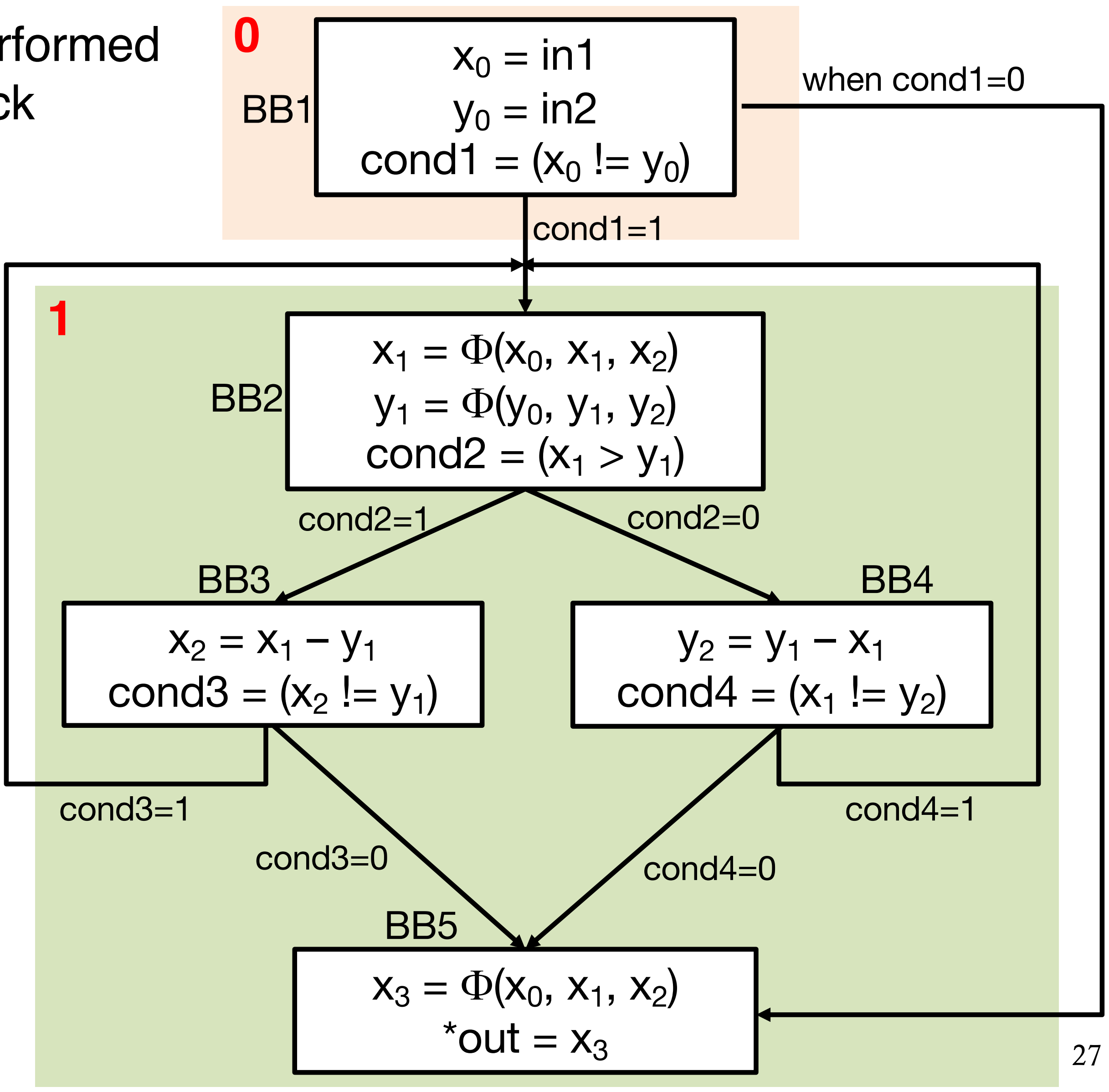
GCD in SSA form

```
x = in1;
y = in2;
while (x != y) {
    if ( x > y )
        x = x – y;
    else y = y – x;
}
*out = x;
```



Interpreting the LP Solution of SDC Scheduling

- Scheduling is performed across basic block boundaries



Operations and Predicates

0

```
$x_0 = in1$
$y_0 = in2$
$cond1 = (x_0 != y_0)$
```

⇨

```
$x_0 = in1$
$y_0 = in2$
$cond1 = (x_0 != y_0)$
```

Add predicates for conditionally executed operations in each state

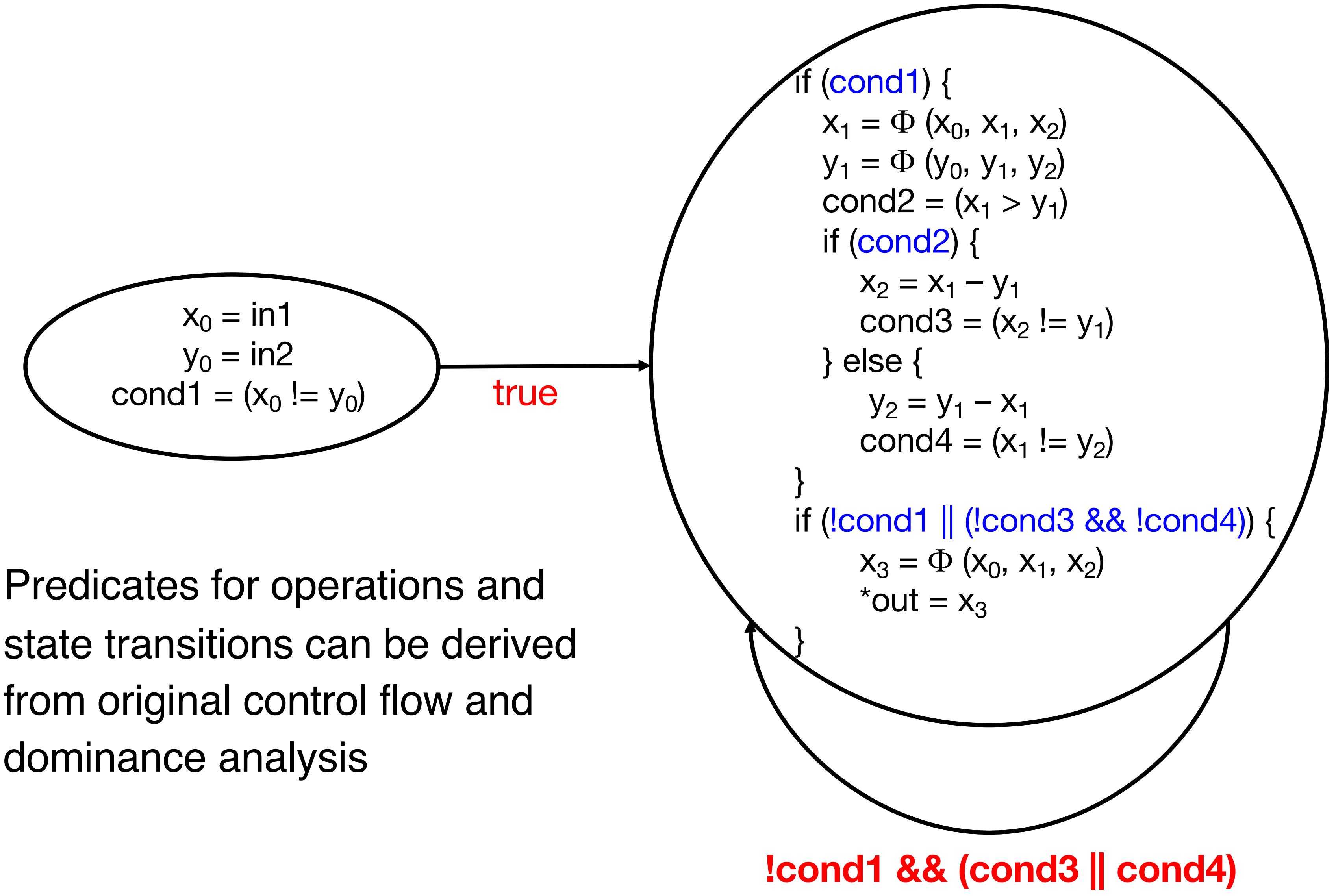
1

```
$x_1 = \Phi (x_0, x_1, x_2)$
$y_1 = \Phi (y_0, y_1, y_2)$
$cond2 = (x_1 > y_1)$
$x_2 = x_1 - y_1$
$cond3 = (x_2 != y_1)$
$y_2 = y_1 - x_1$
$cond4 = (x_1 != y_2)$
$x_3 = \Phi (x_0, x_1, x_2)$
$*out = x_3$
```

⇨

```
if (cond1) {
   $x_1 = \Phi (x_0, x_1, x_2)$
   $y_1 = \Phi (y_0, y_1, y_2)$
   $cond2 = (x_1 > y_1)$
   if (cond2) {
       $x_2 = x_1 - y_1$
       $cond3 = (x_2 != y_1)$
   } else {
       $y_2 = y_1 - x_1$
       $cond4 = (x_1 != y_2)$
}
if (!cond1 || (!cond3 && !cond4)) {
       $x_3 = \Phi (x_0, x_1, x_2)$
       $*out = x_3$
}
```

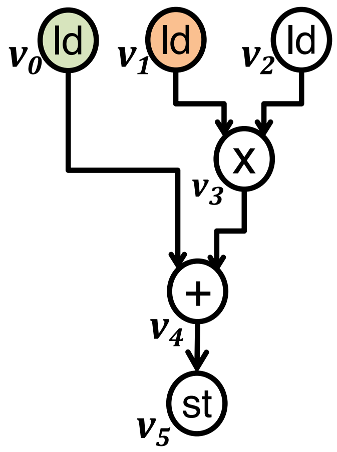
Deriving State Transition Graph (STG)



```
if (cond1) {
   $x_1 = \Phi (x_0, x_1, x_2)$
   $y_1 = \Phi (y_0, y_1, y_2)$
   cond2 = ($x_1 > y_1$)
   if (cond2) {
      $x_2 = x_1 - y_1$
      cond3 = ($x_2$ != $y_1$)
   } else {
      $y_2 = y_1 - x_1$
      cond4 = ($x_1$ != $y_2$)
}
if (!cond1 || (!cond3 && !cond4)) {
      $x_3 = \Phi (x_0, x_1, x_2)$
      *out = $x_3$
}
```

Predicates for operations and state transitions can be derived from original control flow and dominance analysis

*Exact Encoding of Resource Constraints



Two read ports only!

Load operations must be serialized $\longrightarrow$

(**NP-Hard** in general)

	Port1	Port2	DSP
cycle=1	v_0	v_2	
cycle=2	v_1		v_3
cycle=3		v_5	v_4

OR

Port1	Port2	DSP
v_1	v_2	v_3
v_0	v_5	v_4

$$\boxed{s_0 - s_1 \neq 0}$$

$$s_0 - s_1 \leq -1$$

OR

$$s_1 - s_0 \leq -1$$

Difficult to exactly encode resource constraints in the strict SDC form

Using Boolean formulas instead

$R_{0,1}$ Resource sharing variable
v_0 and v_1 share the same port?

$\Downarrow$

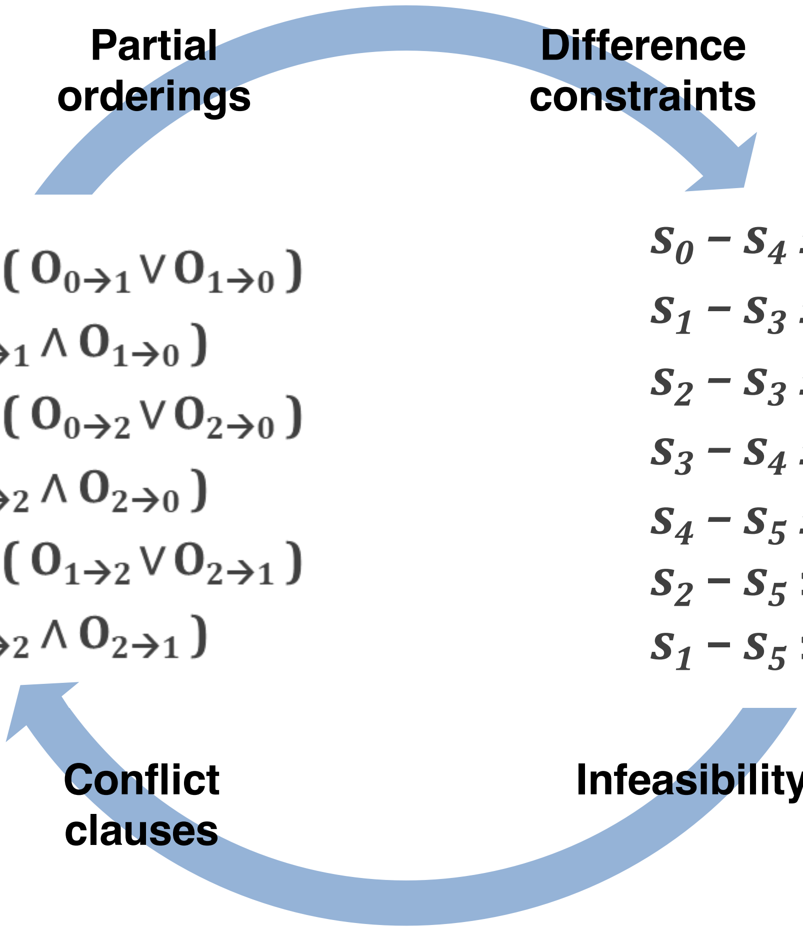
$O_{0\to1}$ Ordering variable
v_0 scheduled before v_1 ?

$\vee$

$O_{1\to0}$ v_1 scheduled before v_0 ?

Note: $R_{0,1} \rightarrow (O_{0\to1} \vee O_{1\to0})$ reads "$R_{0,1}$ implies $O_{0\to1}$ or $O_{1\to0}$"

*SDS: Exact and Practically Scalable Scheduling with SDC and SAT



<u>SAT</u>

Resource Constraints

$R_{01} \rightarrow (O_{0\rightarrow1} \vee O_{1\rightarrow0})$

$\neg(O_{0\rightarrow1} \wedge O_{1\rightarrow0})$

$R_{02} \rightarrow (O_{0\rightarrow2} \vee O_{2\rightarrow0})$

$\neg(O_{0\rightarrow2} \wedge O_{2\rightarrow0})$

$R_{12} \rightarrow (O_{1\rightarrow2} \vee O_{2\rightarrow1})$

$\neg(O_{1\rightarrow2} \wedge O_{2\rightarrow1})$

$s_0 - s_4 \leq 0$

$s_1 - s_3 \leq 0$

$s_2 - s_3 \leq 0$

$s_3 - s_4 \leq 0$

$s_4 - s_5 \leq 0$

$s_2 - s_5 \leq -1$

$s_1 - s_5 \leq -1$

<u>SDC</u>

Timing Constraints

Conflict based search

~1M variables
>1M clauses

Conflict-driven learning

Graph based feasibility checking

Polynomial time

[S. Dai, G. Liu, and Z. Zhang, FPGA 2018]

Scheduling Summary

- ILP
  - Exact, but exponential worst-case runtime

- Hu's algorithm
  - Optimal and polynomial
  - Only works in very restricted cases

- List scheduling
  - Extension to Hu's for general cases
  - Greedy (fast) but suboptimal

- SDC-based scheduling
  - A versatile heuristic based on LP formulation with different constraints
  - Amenable to global optimization

Next Lecture

- Resource sharing
- Pipelining concepts

Acknowledgements

- These slides contain/adapt materials developed by
  - Ryan Kastner (UCSD)