1. Introduction

Your first programming assignment is a warmup designed to give you experience with two important aspects of computer systems programming: software design and software testing. In this assignment, you will leverage the basic concepts from lecture, ranging from variables and operators to conditional and iteration statements. More advanced concepts such as recursion will also play a key role in optimizing the performance of your code.

You will implement the square root and power math functions twice: first using a simple but naive implementation, and then using a more sophisticated algorithm that can potentially improve performance by an order of magnitude. Throughout the assignment, you will carefully design your code to be maintainable (“Will somebody else be able to understand my code and extend it?”). We will leverage the CMake/CTest framework for unit testing, TravisCI for continuous integration testing, and Codecov.io for code coverage analysis.

After your code is functional and tested, you will write a three-page report that provides a detailed deep-dive discussion on one implementation, qualitatively evaluates trade-offs across four implementations, and quantitatively evaluates the performance across six implementations. You should consult the programming assignment assessment rubric for more information about the expectations for all programming assignments and how they will be assessed. While the final code and report are all due at the end of the assignment, we also require meeting an incremental milestone in this PA. Requirements specific to this PA for the incremental milestone and the final report are described at the end of this handout.

This handout assumes that you have read and understand the course tutorials and that you have attended the discussion sections. To get started, log in to an ecelinux server, source the setup script, and clone your individual remote repository from GitHub:

```bash
% source setup-ece2400.sh
% mkdir -p ${HOME}/ece2400
% cd ${HOME}/ece2400
% git clone git@github.com:cornell-ece2400/netid
% cd ${HOME}/ece2400/netid/pa1-math
% tree
```

Where `netid` should be replaced with your NetID. You can both pull and push to your individual remote repository. You should never fork your individual remote repository! If you need to work in isolation then use a branch within your individual remote repository. If you have already cloned your individual remote repository, then use git pull to ensure you have any recent updates before working on your programming assignment.
% cd ${HOME}/ece2400/netid/pa1-math
% git pull
% tree

For this assignment, you will work in the pa1-math subproject. The programming assignment includes the following files:

- CMakeLists.txt
  - CMake configuration script to generate Makefile
- src/ece2400-stdlib.c
  - Source code for course standard library
- src/ece2400-stdlib.h
  - Header file for course standard library
- src/sqrt-iter.c
  - Source code for iterative sqrt
- src/sqrt-iter.h
  - Header file for iterative sqrt
- src/sqrt-iter-adhoc.c
  - Ad-hoc test program for iterative sqrt
- src/sqrt-recur.c
  - Source code for recursive sqrt
- src/sqrt-recur.h
  - Header file for recursive sqrt
- src/sqrt-recur-adhoc.c
  - Ad-hoc test program for recursive sqrt
- src/pow-iter.c
  - Source code for iterative pow
- src/pow-iter.h
  - Header file for iterative pow
- src/pow-iter-adhoc.c
  - Ad-hoc test program for iterative pow
- src/pow-recur.c
  - Source code for recursive pow
- src/pow-recur.h
  - Header file for recursive pow
- src/pow-recur-adhoc.c
  - Ad-hoc test program for recursive pow
- eval/sqrt-iter-eval.c
  - Evaluation program for iterative sqrt
- eval/sqrt-recur-eval.c
  - Evaluation program for recursive sqrt
- eval/sqrt-std-eval.c
  - Evaluation program for standard sqrt
- eval/pow-iter-eval.c
  - Evaluation program for iterative pow
- eval/pow-recur-eval.c
  - Evaluation program for recursive pow
- eval/pow-std-eval.c
  - Evaluation program for standard pow
- test/sqrt-iter-directed-test.c
  - Directed test cases for iterative sqrt
- test/sqrt-iter-random-test.c
  - Random test cases for iterative sqrt
- test/sqrt-recur-directed-test.c
  - Directed test cases for recursive sqrt
- test/sqrt-recur-random-test.c
  - Random test cases for recursive sqrt
- test/pow-iter-directed-test.c
  - Directed test cases for iterative pow
- test/pow-iter-random-test.c
  - Random test cases for iterative pow
- test/pow-recur-directed-test.c
  - Directed test cases for recursive pow
- test/pow-recur-random-test.c
  - Random test cases for recursive pow
2. Implementation Specifications

In this project, you will be implementing both iterative and recursive algorithms to compute the square root (\texttt{sqrt}) and power (\texttt{pow}) math functions. The algorithms used for the iterative implementations of \texttt{sqrt} and \texttt{pow} are simple but slow. The recursive algorithms are more complex but can potentially improve performance by an order of magnitude compared to their iterative versions. The programming assignment is divided into four steps. Implement and test each step before moving on to the next step.

- Step 1. Implement and test \texttt{sqrt\_iter} function using an iterative algorithm
- Step 2. Implement and test \texttt{pow\_iter} function using an iterative algorithm
- Step 3. Implement and test \texttt{sqrt\_recur} function using a recursive algorithm
- Step 4. Implement and test \texttt{pow\_recur} function using a recursive algorithm

Note that your implementations cannot use anything from the Standard C library except for the \texttt{printf} function defined in \texttt{stdio.h}, the MIN/MAX macros defined in \texttt{limits.h}, the \NULL and \texttt{size\_t} macros defined in \texttt{stddef.h}, and the \texttt{assert} macro defined in \texttt{assert.h}.

2.1. Square Root Function

The \texttt{sqrt} function takes one input argument \(x\) and returns its square root (i.e., \(\sqrt{x}\)). The corresponding C function has the following function signature:

\[
\text{int } \text{sqrt}( \text{int } x )
\]

Notice that the variant of \texttt{sqrt} that we will use in this assignment takes an integer input and returns another integer. The return value is the square root of \(x\) rounded down to the nearest integer. For example, calling \texttt{sqrt(5)} will return 2. If \(x\) is a negative value, the \texttt{sqrt} function must return -1 to report an invalid input. Your implementations should work correctly when the input is zero and when the input is any valid positive integer. Your implementations should avoid any floating point arithmetic since this can introduce unnecessary performance overhead.

**Iterative \texttt{sqrt} Implementation**

The iterative implementation of the \texttt{sqrt} function should be implemented using an iteration statement. Let \(i\) range from zero to \(x\). For each \(i\), compute \(i \times i\) and compare the result with \(x\). If \(i \times i\) is smaller than \(x\), then \(i\) is less than the square root of \(x\). If \(i \times i\) is larger than \(x\), then \(i\) is greater than the square root of \(x\). By gradually checking all values of \(i\), you will be able to find the square root of \(x\) rounded down to the nearest integer. Write your iterative implementation for \texttt{sqrt} inside of \texttt{src/sqrt-iter.c}.

**Recursive \texttt{sqrt} Implementation**

The iterative implementation of \texttt{sqrt} is particularly slow when \(x\) is large because: (1) the computer executes multiplication operations more slowly compared to simpler operations (e.g., addition, subtraction); and (2) the number of multiply operations increases linearly with \(\sqrt{x}\) since we are doing an exhaustive search. We can use a more sophisticated search to reduce the number of multiply operations. Consider the situation when \(x\) is 144. We can divide the search space into two ranges:

- Range of integers from 0 to \(\frac{x}{2}\), which is [0, 72] when \(x\) is 144
- Range of integers from \(\frac{x}{2}\) to \(x\), which is [72, 144] when \(x\) is 144
Figure 1: Example of Recursive sqrt Algorithm – The range of integers that can contain the square root is halved at each step.

We can quickly determine which half the square root of $x$ lies in by squaring the midpoint (i.e., $72 \times 72 = 5184$) and comparing to $x$. Observing $5184 > 144$ tells us that our guess of 72 was much too high, so the answer must be in the lower half (i.e., somewhere in the range $[0, 72]$), which is true since we know in this example that the square root is 12. We can continue applying the same approach on the smaller range, dividing the search space into smaller and smaller ranges. Figure 1 illustrates an example execution when $x$ is 144. This approach allows us to quickly “zoom in” on the square root of $x$. We can capture this algorithm iteratively, but a recursive solution is also possible and may be more elegant and concise. The general approach of repeatedly halving the search space is known as a binary search. We will learn more about this class of algorithms in the future. Write your recursive implementation for sqrt inside of src/sqrt-recur.c. You may add additional helper functions inside this file as needed.

2.2. Power Function

The pow function takes two input arguments: (1) the base value ($b$), and (2) the exponent value ($e$). The function returns the base raised to the power of the exponent (i.e., $b^e$). Specifically, the corresponding C function has the following function signature:

```
double pow( double base, int exponent )
```

Notice that base is a floating-point double (i.e., a real number), exponent is an integer, and the returned value is also a floating-point double. A basic implementation of the pow function expressed
in a mathematical formula is:

\[ b^e = \begin{cases} 
1 & \text{if } e = 0 \\
 b \times b \times \cdots \times b & \text{if } e > 0 \\
1/(b \times b \times \cdots \times b) & \text{if } e < 0 
\end{cases} \]

In this assignment, we will allow certain cases in the implementation of \( \text{pow} \) to remain undefined:

- The result for \( 0^e \) where \( e \leq 0 \) is undefined
- The result when \( b < -1000 \) or when \( b > 1000 \) is undefined
- The result when \( e < -400 \) or when \( e > 400 \) is undefined
- The result when \( b^e > 10^{300} \) is undefined
- The result when \( b^e < -10^{300} \) is undefined

We will not test these cases when we grade the assignment. Students also do not need to test these cases, nor handle these special cases explicitly in their implementation.

**Iterative \( \text{pow} \) Implementation**

Your iterative implementation of the \( \text{pow} \) function should directly correspond to the above definition and should use an iteration statement with a few carefully chosen conditional statements. Write your iterative implementation for \( \text{pow} \) inside of \( \text{src/pow-iter.c} \).

**Recursive \( \text{pow} \) Implementation**

The iterative implementation of \( \text{pow} \) is particularly slow when the exponent is large because: (1) as mentioned above, the computer executes multiplication operations more slowly compared to simpler operations (e.g., addition, subtraction); and (2) the number of multiply operations increases linearly with \( e \). We can exploit structure in the iterative implementation to reduce the required number of multiply operations. Consider the following example.

\[ 3^8 = 3 \times 3 \times 3 \times 3 \times 3 \times 3 \times 3 \times 3 \]

We can reduce the number of multiply operations by reusing intermediate results.

\[
\begin{align*}
a &= 3 \times 3 \\
b &= a \times a \\
c &= b \times b
\end{align*}
\]

We first calculate \( 3^2 = 3 \times 3 \), and we can then reuse \( 3^2 \) to calculate \( 3^4 = (3^2)^2 \), and we can then reuse \( 3^4 \) to calculate \( 3^8 = ((3^2)^2)^2 \). The iterative implementation would require seven multiply operations, while the recursive implementation requires only three multiply operations. Here is the recursive implementation expressed as a mathematical formula:

\[ b^e = \begin{cases} 
1 & \text{if } e = 0 \\
(b^2)^{e/2} & \text{if } e > 0 \text{ and } e \text{ is even} \\
b \times b^{e-1} & \text{if } e > 0 \text{ and } e \text{ is odd} \\
1/(b^{|e|}) & \text{if } e < 0 
\end{cases} \]
Notice that this approach for computing \texttt{pow} has multiple recursive cases. Although an iterative solution using loops is possible, we can more elegantly and concisely capture this algorithm using a recursive helper function. Write your recursive implementation for \texttt{pow} inside of \texttt{src/pow-recur.c}. You may add additional helper functions inside this file as needed.

3. Testing Strategy

You are responsible for developing an effective testing strategy to ensure all implementations are correct. Writing tests is one of the most important and challenging aspects of software programming. Software engineers often spend far more time implementing tests than they do implementing the actual program.

Note that while there are limitations on what you can use from the Standard C library in your implementations there are no limitations on what you can use from the Standard C library in your testing strategy. You should feel free to use the Standard C library in your golden reference models and/or for random testing.

3.1. Ad-hoc Testing

To help students start testing, we provide an ad-hoc test program for each implementation (e.g., \texttt{src/sqrt-iter-adhoc.c}). Students are encouraged to start compiling and running these ad-hoc test programs directly in the \texttt{src} directory without using any build or test frameworks (e.g., CMake or Make).

You can build and run the given ad-hoc test programs like this:

```
% cd ${HOME}/ece2400/netid/pa1-math/src
% gcc -Wall -o sqrt-iter-adhoc ece2400-stdlib.c sqrt-iter.c sqrt-iter-adhoc.c
% ./sqrt-iter-adhoc
% gcc -Wall -o sqrt-recur-adhoc ece2400-stdlib.c sqrt-recur.c sqrt-recur-adhoc.c
% ./sqrt-recur-adhoc
% gcc -Wall -o pow-iter-adhoc ece2400-stdlib.c pow-iter.c pow-iter-adhoc.c
% ./pow-iter-adhoc
% gcc -Wall -o pow-recur-adhoc ece2400-stdlib.c pow-recur.c pow-recur-adhoc.c
% ./pow-recur-adhoc
```

The \texttt{--Wall} flags will ensure that \texttt{gcc} reports all warnings.

3.2. Systematic Unit Testing

While ad-hoc test programs help you quickly see results of your implementations, these ad-hoc tests are often too simple to cover most scenarios. We need a systematic and automatic unit testing strategy to hopefully test all possible scenarios efficiently.

In this course, we are using CMake/CTest as our build and test automation framework. For each implementation, we provide a directed test program that should include several test cases to target different categories and a random test program that should test that your implementation works for
random inputs. **We only provide a very few directed tests and no random tests. You must add many more directed and random tests to thoroughly test your implementations!**

A directed test case involves manually pre-computing the output for a specific set of inputs, and then verifying that your implementation produces this desired output. Start by writing as many directed test cases as you can for some simple and more complex inputs. As you design your implementations, pay careful attention to corner cases and unexpected inputs (e.g., negative inputs) that break the functionality of your code. When you encounter such a case, capture the situation with a directed test case and verify your implementation now passes that test case. Carefully read the implementation specification (i.e., the inputs, the outputs, and the behavior), so you know how your program should respond in all possible scenarios. Convince yourself that your implementations are **robust** by carefully developing a testing strategy.

In addition to writing directed tests, you should also add random tests to increase your confidence in the correctness of your implementation. You can randomly generate inputs using the `rand` function in the standard C library (include `stdlib.h`). Use the `srand` function to initialize the random seed to a deterministic value to ensure your random tests are repeatable. You can use the `sqrt` and `pow` functions in the standard C library (include `math.h`) as golden reference models to generate correct reference outputs which you can then compare to the results from your own implementations. Note that you are not allowed to use the `sqrt` and `pow` functions in the standard C library for your implementation, only for testing.

When testing the `pow` function, note that the result can vary by very small amounts because of precision errors that build up as the computer performs arithmetic on real numbers. This is commonly solved by comparing relative amounts (i.e., checking that the two numbers are within at least 99.99% of each other). Example tests written in this fashion have been provided for you in `tests/pow-iter-directed-tests.c` and `tests/pow-recur-directed-tests.c`. If precision error becomes a problem, please compare numbers relatively.

Before running the tests you need to create a separate build directory and use `cmake` to create the Makefile like this:

```bash
% cd ${HOME}/ece2400/netid/pa1-math
% mkdir -p build
% cd build
% cmake ..
```

Now you can build and run all unit tests for all implementations like this:

```bash
% cd ${HOME}/ece2400/netid/pa1-math/build
% make check
```

If you are failing a test program, then you can “zoom in” and run all of the unit tests for a single test program (e.g., directed tests for `sqrt-iter`) like this:

```bash
% cd ${HOME}/ece2400/netid/pa1-math/build
% make sqrt-iter-directed-test
% ./sqrt-iter-directed-test
```

You can then “zoom in” to a specific test case by passing in the index of that test case like this:

```bash
% cd ${HOME}/ece2400/netid/pa1-math/build
% make sqrt-iter-directed-test
```
3.3. Code Coverage

After your implementations pass all unit tests, you can evaluate how effective your test suite is by measuring its code coverage. The code coverage will tell you how much of your source code your test suite executed during your unit testing. The higher the code coverage is, the less likely some bugs have not been detected. You can run the code coverage like this:

```bash
% cd ${HOME}/ece2400/netid/pa1-math
% rm -rf build-coverage
% mkdir -p build-coverage
% cd build-coverage
% cmake ..
% make check
% make coverage
% elinks coverage-html/index.html
```

Note that these code coverage results will reflect all prior runs of the test and evaluation programs in the build directory. That is why in the above example, we do a fresh build in a separate build-coverage build directory.

Code coverage is just one more piece of evidence you can use to make a compelling case for the correct functionality of your implementations. It is not required that students achieve 100% code coverage. It is far more important that students simply use code coverage as a way to guide their test-driven design than to overly focus on the specific code coverage number.

4. Evaluation

Once you have tested the functionality of the iterative and recursive implementations, you can then start to evaluate the performance of these implementations. We provide you a performance analysis harness for each implementation. In addition, we also provide you two performance analysis harnesses for the `sqrt` and `pow` functions provided in the standard `math` library. You can build the evaluation programs like this:

```bash
% cd ${HOME}/ece2400/netid/pa1-math
% rm -rf build-eval
% mkdir -p build-eval
% cd build-eval
% cmake -DCMAKE_BUILD_TYPE=eval ..
% make eval
```

Note how we are working in a separate build-eval build directory, and that we are using the `-DCMAKE_BUILD_TYPE=eval` command line option to the `cmake` script. This tells the build system to create optimized executables without any extra debugging information. **You must do your quantitative evaluation using an eval build. Using a debug build for evaluation produces meaningless results.**

To run an evaluation, you simply specify the inputs on the command line. For example, the following runs an evaluation for one of the `sqrt` implementations to find the square root of 100.
The following runs an evaluation for one of the pow implementations for a base of 2 and an exponent of 5.

```
% cd ${HOME}/ece2400/netid/pa1-math/build-eval
% make eval
% ./pow-iter-eval 2 5
```

The evaluation programs apply your math functions to the input you specify at the command line in a loop and report the total wall-clock runtime. This will enable you to compare the performance between your iterative algorithms, recursive algorithms, and the implementations provided in the standard math library. The evaluation programs also ensure that your implementations are producing the correct results, however, you should not use the evaluation programs for testing. If your implementations fail during the evaluation, then your testing strategy is insufficient. You must add more unit tests to effectively test your program before returning to performance evaluation.

You should quantitatively evaluate all six evaluations for a range of values. We suggest evaluating the sqrt implementations from zero to one million with many intermediate points. We suggest evaluating the pow implementations using a base of 4.2 and exponents starting at 0 and ending at 300 with many intermediate points. Record all of this performance data and create two plots. The first plot should have the input to sqrt on the x-axis and the wall-clock runtime on the y-axis. Plot a line for each of the three implementations of sqrt. The second plot should have the exponent used with pow on the x-axis and the wall-clock runtime on the y-axis. (Remember that all evaluation for pow should use 4.2 for the base.) Plot a line for each of the three implementations of pow. **Students are required to include these two plots in their report, and to discuss these results in the quantitative evaluation section.** Ensure your plots are easy to read with a legend, reasonable font sizes, and appropriate colors/markers for black-and-white printing.

5. **Milestone and Report**

This section includes critical information about the incremental milestone and the final report specific to this PA.

5.1. **Incremental Milestone**

The programming assignment assessment rubric provides general details about the requirements for the incremental milestone. While the final code and report are all due at the end of the assignment, we also require meeting an incremental milestone of completing steps 1 and 2 (i.e., the iterative implementations of both sqrt and pow) and then pushing the corresponding code to GitHub on the date specified by the instructor. More specifically to meet the incremental milestone of this PA, you are expected to:

- Complete the iterative implementation of sqrt
- Complete the iterative implementation of pow
- Pass all given directed tests for both implementations
- Thoroughly test both implementations by adding your own directed and random tests

Here is how we will be testing your milestone:
% cd ${HOME}/ece2400/netid/pa1-math
% mkdir -p build
% cd build
% cmake ..
% make check-milestone

5.2. Final Report

The programming assignment assessment rubric provides general details about the requirements for the final report. You must actually read the rubric to ensure you know what goes in your report and how to format it. For this PA, we require you to discuss your recursive implementation of square root in the implementation deep dive section. For this PA, we also require you to include two performance plots which you reference in the quantitative evaluation section of your report. These performance plots should show the wall-clock runtime vs. input. See the previous section of this handout for more information about these plots.

Acknowledgments

This programming assignment was created by Christopher Batten, Jose Martínéz, Christopher Torng, Xiaodong Wang, Shuang Chen, Shunning Jiang, Tuan Ta, Yanghui Ou, and Peitian Pan as part of the ECE 2400 Computer Systems Programming course at Cornell University.