Animation and Quaternions

1

Pixar's Luxo Jr.

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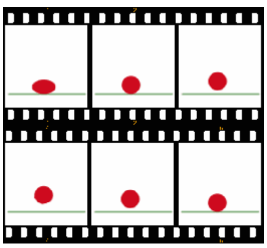
A New Dimension - Time



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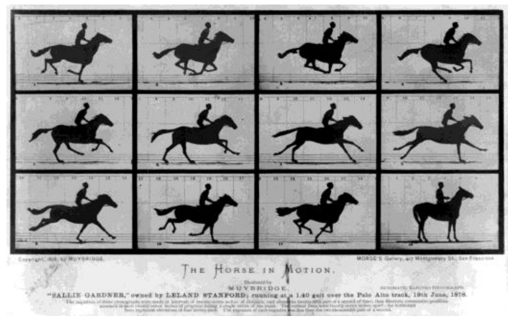
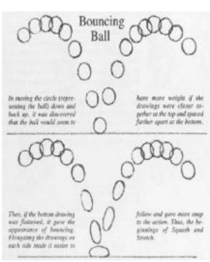
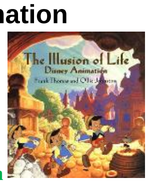


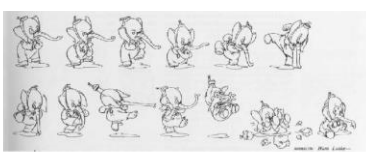
12 Principles of Traditional Animation



Anticipation



Squash/Stretch



Secondary Action

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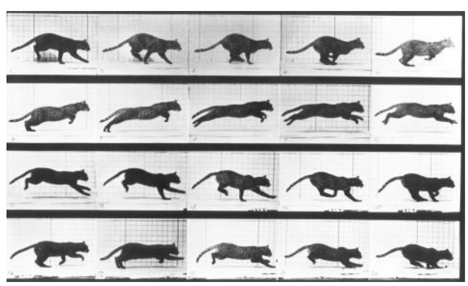
Specifying Animation



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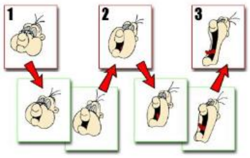
Keyframes



Automatically fill in motion between these points in time

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Manual Tweening

Tweening = Interpolation

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VRML Examples

Fred

Floops

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Animation in VRML

```
DEF PI PositionInterpolator {
        key [ 0 0.5 1 ]                  t_i
        keyValue [
                0 0 50,                  (x(t_i) y(t_i) z(t_i))
                0 2 50,
                0 0 50
]}

DEF OI OrientationInterpolator {
        key [ 0 1 ]                      t_i
        keyValue [
                0 0 1 0,
                0 -1 0 1.570796,         (n_x(t_i) n_y(t_i) n_z(t_i) θ(t_i))   quaternion
]}
```

rotation without quaternions   rotation with quaternions

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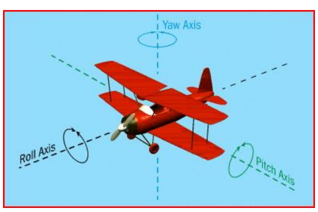
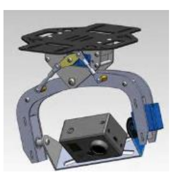
Three Degrees of Freedom





Gimball

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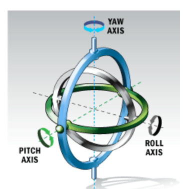
Orientation and Euler Angles

- The orientation of any 3D object may be described by three Euler angles: successive rotations (relative to an initial orientation) around three axes:
  - *x*-roll: rotation around *x* axis.
  - *y*-roll: rotation around *y* axis.
  - *z*-roll: rotation around *z* axis.
- The order of rotation application is important, as rotations do not commute.



An obvious way to interpolate two orientations is to interpolate their Euler angles.

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Rotation by Euler Angles

$$R_x(\alpha)=\begin{bmatrix}1&0&0&0\\0&\cos\alpha&\sin\alpha&0\\0&-\sin\alpha&\cos\alpha&0\\0&0&0&1\end{bmatrix}\qquad R_y(\beta)=\begin{bmatrix}\cos\beta&0&-\sin\beta&0\\0&1&0&0\\\sin\beta&0&\cos\beta&0\\0&0&0&1\end{bmatrix}$$

$$R_z(\gamma)=\begin{bmatrix}\cos\gamma&\sin\gamma&0&0\\-\sin\gamma&\cos\gamma&0&0\\0&0&1&0\\0&0&0&1\end{bmatrix}$$

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Gimbal Lock

$$R(\alpha,\beta,\gamma)=R_z\cdot R_y\cdot R_x=$$

$$\begin{bmatrix}\cos\gamma\cos\beta & \sin\gamma\cos\alpha+\cos\gamma\sin\beta\sin\alpha & \sin\gamma\sin\alpha-\cos\gamma\sin\beta\cos\alpha & 0\\ -\sin\gamma\cos\beta & \cos\gamma\cos\alpha-\sin\gamma\sin\beta\sin\alpha & \cos\gamma\sin\alpha+\sin\gamma\sin\beta\cos\alpha & 0\\ \sin\beta & -\cos\beta\sin\alpha & \cos\beta\cos\alpha & 0\\ 0&0&0&1\end{bmatrix}$$

When $\beta=\frac{\pi}{2}$ this is:

$$\begin{bmatrix}0 & \sin\gamma\cos\alpha+\cos\gamma\sin\alpha & \sin\gamma\sin\alpha-\cos\gamma\cos\alpha & 0\\ 0 & \cos\gamma\cos\alpha-\sin\gamma\sin\alpha & \cos\gamma\sin\alpha+\sin\gamma\cos\alpha & 0\\ 1&0&0&0\\0&0&0&1\end{bmatrix}=\begin{bmatrix}0&\sin\theta&-\cos\theta&0\\0&\cos\theta&\sin\theta&0\\1&0&0&0\\0&0&0&1\end{bmatrix}$$

where $\theta=\alpha+\gamma$, so one degree of freedom is lost.

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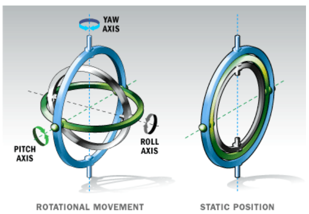

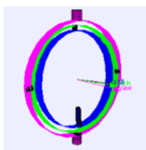

Gimbal Lock

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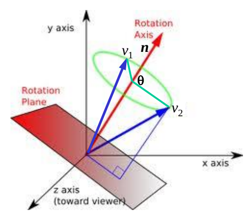
General Rotations

Euler, 1756

Theorem: (Euler): In 3D, any displacement of a rigid body such that a point on the rigid body remains fixed, is equivalent to a single rotation about some axis that runs through the fixed point.



Equivalent: In 3D space, any two Cartesian coordinate systems with a common origin are related by a rotation about some fixed axis.

Conclusion: Any 3D orientation may be described by four parameters: angle θ and unit axis $n=(n_x,n_y,n_z)$.

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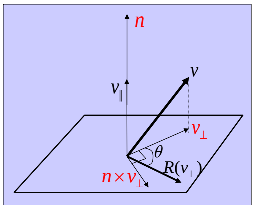
General Rotations (cont'd)

$$\begin{aligned}R(v)&=R(v_{\parallel}+v_{\perp})\\&=R(v_{\parallel})+R(v_{\perp})\\&=v_{\parallel}+v_{\perp}\cos\theta\\&\quad+(n\times v)\sin\theta\\&=(n\cdot v)n+(v-(n\cdot v)n)\cos\theta\\&\quad+(n\times v)\sin\theta\\&=v\cos\theta+n(n\cdot v)(1-\cos\theta)\\&\quad+(n\times v)\sin\theta\end{aligned}$$



$v_{\parallel}=(n\cdot v)n \qquad v_{\perp}=v-v_{\parallel}$

$$\begin{aligned}R(v_{\perp})&=v_{\perp}\cos\theta+(n\times v_{\perp})\sin\theta\\&=v_{\perp}\cos\theta+(n\times v)\sin\theta\end{aligned}$$

$R(v_{\parallel})=v_{\parallel}$

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Quaternions

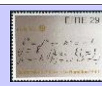
Hamilton 1843

Definition: A *quaternion* is a quadruple $q=[s,v]$, where s is a scalar, and v a three-dimensional vector.

The quaternions form a non-commutative group under the multiplication rule:

$$\begin{aligned}q_1\cdot q_2&=[s_1,v_1]\cdot[s_2,v_2]\\&=[s_1s_2-v_1\cdot v_2,\ s_1v_2+s_2v_1+v_1\times v_2]\end{aligned}$$

Equivalent to:



$$q=s+iv_x+jv_y+kv_z$$

where $i^2=j^2=k^2=-1, ij=k, ji=-k$

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Quaternions (cont'd)

The *conjugate* of $q=[s,v]$ is
$$\bar{q}=[s,-v]$$

$$\overline{q_1 \cdot q_2} = \overline{q_2} \cdot \overline{q_1}$$

The *norm* of $q=[s,v]$ is
$$\|q\|^2 = q \cdot \bar{q} = s^2 + v_x^2 + v_y^2 + v_z^2$$

The *inverse* of $q=[s,v]$ is:
$$q^{-1} = \frac{\bar{q}}{\|q\|^2}$$

Corollary:
$$\|q_1 \cdot q_2\| = \|q_1\| \cdot \|q_2\|$$

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Special Cases

- **Scalars:** [c,(0,0,0)]
- **Complex numbers:** [x,(y,0,0)]
- **3D Vectors:** [0,(x,y,z)]

$$q_1 \cdot q_2 = [s_1, v_1] \cdot [s_2, v_2]$$
$$= [s_1 s_2 - v_1 \cdot v_2,\ s_1 v_2 + s_2 v_1 + v_1 \times v_2]$$

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Rotating with Quaternions

- Rotation by θ around unit direction n may be represented by the unit quaternion
$$q = [\cos\tfrac{\theta}{2}, n\sin\tfrac{\theta}{2}]$$
- The 3D vector $[0,v]$ is rotated by q to:
$$R_q(v) = q^{-1} \cdot v \cdot q = \bar{q} \cdot v \cdot q$$

Since

$$R_q(v) = [\cos\tfrac{\theta}{2}, -n\sin\tfrac{\theta}{2}] \cdot [0,v] \cdot [\cos\tfrac{\theta}{2}, n\sin\tfrac{\theta}{2}]$$
$$= [0, v(\cos^2\tfrac{\theta}{2} - \sin^2\tfrac{\theta}{2}) + 2n(n \cdot v)\sin^2\tfrac{\theta}{2} + 2(n \times v)\cos\tfrac{\theta}{2}\sin\tfrac{\theta}{2}]$$
$$= [0, v\cos\theta + n(n \cdot v)(1-\cos\theta) + (n \times v)\sin\theta]$$

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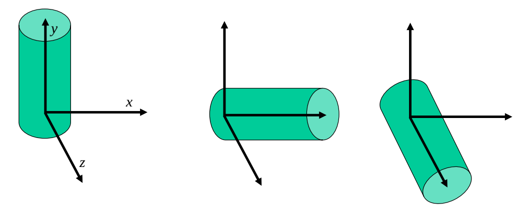
Examples

right hand coordinate system



$n=(0,1,0),\ \theta=0$
$q=(1,0,0,0)$

$n=(0,0,1), \theta=-\pi/2$
$q=(1/\sqrt{2},0,0,-1/\sqrt{2})$

$n=(1,0,0), \theta=\pi/2$
$q=(1/\sqrt{2},1/\sqrt{2},0,0)$

$$q = (\cos\tfrac{\theta}{2}, n\sin\tfrac{\theta}{2}) = (w,x,y,z)$$

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Unit Quaternions

The quaternions used for rotation have only three degrees of freedom. They all lie on the surface of a unit sphere in 4D space, forming a subgroup.

Theorem: A vector *v* is invariant under rotations around an axis through *v*.

Proof: The rotation operator is $q=[s,cv]$ such that $||q||^2=1$.

$$\bar{q} \cdot v \cdot q = [s,-cv] \cdot [0,v] \cdot [s,cv]$$
$$= [s,-cv] \cdot [-cv \cdot v, sv]$$
$$= [0, s^2 v + c^2 (v \cdot v) v]$$
$$= (s^2 + c^2 (v \cdot v))[0,v]$$
$$= \|q\|^2 v$$
$$= v$$

Theorem: Rotation by θ and $-\theta$ in the opposite direction are equivalent.

Proof: $[\cos\tfrac{-\theta}{2}, -n\sin\tfrac{-\theta}{2}] = [\cos\tfrac{\theta}{2}, n\sin\tfrac{\theta}{2}]$

Theorem: Rotation by θ_1 and then by θ_2 (around *n*) is equivalent to rotation by $\theta_1+\theta_2$.

Proof: $(\overline{q_2} \cdot (\overline{q_1} \cdot v \cdot q_1) \cdot q_2) = (\overline{q_2} \cdot \overline{q_1}) \cdot v \cdot (q_1 \cdot q_2)$
$= (\overline{q_1 \cdot q_2}) \cdot v \cdot (q_1 \cdot q_2)$

$[\cos\tfrac{\theta_1}{2}, n\sin\tfrac{\theta_1}{2}] \cdot [\cos\tfrac{\theta_2}{2}, n\sin\tfrac{\theta_2}{2}] =$
$[\cos\tfrac{\theta_1}{2}\cos\tfrac{\theta_2}{2} - \sin\tfrac{\theta_1}{2}\sin\tfrac{\theta_2}{2},$
$n(\sin\tfrac{\theta_1}{2}\cos\tfrac{\theta_2}{2} + \cos\tfrac{\theta_1}{2}\sin\tfrac{\theta_2}{2}] =$
$[\cos\tfrac{\theta_1+\theta_2}{2}, n\sin\tfrac{\theta_1+\theta_2}{2}]$

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Quaternion to Rotation Matrix (column vectors)

Rotation by unit-length quaternion:

$$q = [\cos\tfrac{\theta}{2}, n\sin\tfrac{\theta}{2}] = [w,x,y,z]$$
$$R_q([0,v]) = \bar{q} \cdot [0,v] \cdot q =$$
$$v\cos\theta + n(n \cdot v)(1-\cos\theta) + (n \times v)\sin\theta$$

The first row of the equivalent rotation matrix:

$$R_q([0,v])_x = v_x\cos\theta + n_x(n_x v_x + n_y v_y + n_z v_z)(1-\cos\theta) + (n_y v_z - n_z v_y)\sin\theta$$

$$M_{11} = \cos\theta + n_x^2(1-\cos\theta)$$
$$M_{12} = n_x n_y (1-\cos\theta) - n_z\sin\theta$$
$$M_{13} = n_x n_z (1-\cos\theta) + n_y\sin\theta$$

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Quaternion to Rotation Matrix (column vectors)

$$q=[\cos\tfrac{\theta}{2}, n\sin\tfrac{\theta}{2}]=[w,x,y,z]$$

$$\begin{aligned}M_{11}&=\cos\theta+n_x^2(1-\cos\theta)\\&=1-2\sin^2\tfrac{\theta}{2}+n_x^2(2\sin^2\tfrac{\theta}{2})\\&=1-2(n_x^2+n_y^2+n_z^2)\sin^2\tfrac{\theta}{2}+n_x^2(2\sin^2\tfrac{\theta}{2})\\&=1-2(n_y^2+n_z^2)\sin^2\tfrac{\theta}{2}\\&=1-2y^2-2z^2\end{aligned}$$

$$M=\begin{pmatrix}1-2y^2-2z^2 & 2xy-2wz & 2xz+2wy & 0\\ 2xy+2wz & 1-2x^2-2z^2 & 2yz-2wx & 0\\ 2xz-2wy & 2yz+2wx & 1-2x^2-2y^2 & 0\\ 0&0&0&1\end{pmatrix}$$

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Rotation Matrix to Quaternion

Given rotation matrix

$$M=\begin{pmatrix}M_{11}&M_{12}&M_{13}&0\\M_{21}&M_{22}&M_{23}&0\\M_{31}&M_{32}&M_{33}&0\\0&0&0&1\end{pmatrix}$$

The equivalent quaternion is $q=[w,x,y,z]$

$$M=\begin{pmatrix}1-2y^2-2z^2 & 2xy-2wz & 2xz+2wy & 0\\ 2xy+2wz & 1-2x^2-2z^2 & 2yz-2wx & 0\\ 2xz-2wy & 2yz+2wx & 1-2x^2-2y^2 & 0\\ 0&0&0&1\end{pmatrix}$$

$$M_{11}+M_{22}+M_{33}+1=4-4(y^2+z^2+x^2)=4w^2$$

$$w=\frac{\sqrt{M_{11}+M_{22}+M_{33}+1}}{2}$$

$$M_{32}-M_{23}=4wx \;\Rightarrow\; x=\frac{M_{32}-M_{23}}{4w}$$

$$M_{13}-M_{31}=4wy \;\Rightarrow\; y=\frac{M_{13}-M_{31}}{4w}$$

$$M_{21}-M_{12}=4wz \;\Rightarrow\; z=\frac{M_{21}-M_{12}}{4w}$$

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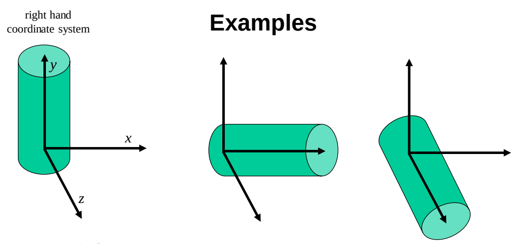
Examples



$n=(0,1,0),\ \theta=0$

$q=(1,0,0,0)$

$$M=\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}$$

$n=(0,0,1), \theta=-\pi/2$

$q=(1/\sqrt{2},0,0,-1/\sqrt{2})$

$$M=\begin{pmatrix}0&1&0\\-1&0&0\\0&0&1\end{pmatrix}$$

$n=(1,0,0), \theta=\pi/2$

$q=(1/\sqrt{2},1/\sqrt{2},0,0)$

$$M=\begin{pmatrix}1&0&0\\0&0&-1\\0&1&0\end{pmatrix}$$

$$q=(\cos\tfrac{\theta}{2}, n\sin\tfrac{\theta}{2})=(w,x,y,z)$$

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Prove Using Quaternions and Matrices

- The quaternion i=(0,1,0,0) is rotation by π around the x axis.
- The quaternion j=(0,0,1,0) is rotation by π around the y axis.
- The quaternion k=(0,0,0,1) is rotation by π around the z axis.

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Animation in VRML

```
DEF PI PositionInterpolator {
        key [ 0 0.5 1 ]              t_i
        keyValue [
                0 0 50,              (x(t_i) y(t_i) z(t_i))
                0 2 50,
                0 0 50
]}

DEF OI OrientationInterpolator {
        key [ 0 1 ]                  t_i
        keyValue [
                0 0 1 0,
                0 -1 0 1.570796,     (n_x(t_i) n_y(t_i) n_z(t_i) θ(t_i))   quaternion
]}
```

rotation without quaternions   rotation with quaternions

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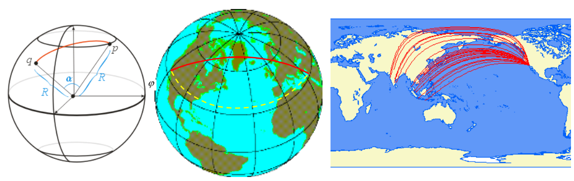
Geodesics

- A *great circle* is a circle on the surface of a sphere that has the same circumference as the sphere, dividing the sphere into two equal hemispheres.
- A great circle is a spherical *geodesic* – the shortest path between two points on a sphere.



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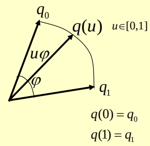
Interpolating Two Orientations

- Orientations may be interpolated by interpolating their respective quaternions.
- Quaternions on a 4D sphere may be interpolated along a geodesic (the shortest path between two points on a sphere).
- For any $u \in [0,1]$, writing $q(u) = \alpha(u)q_0+\beta(u)q_1$, and solving the following system of equations for $\alpha(u)$ and $\beta(u)$:

$$\cos\varphi = \|q_0\|\|q_1\|\cos\varphi = < q_0, q_1 >$$
$$1 = \|q(u)\|^2 = \alpha(u)^2 + \beta(u)^2 + 2\alpha(u)\beta(u) < q_0, q_1 >$$
$$\cos(u\varphi) = < q_0, q(u) > = \alpha(u) + \beta(u) < q_0, q_1 >$$



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$$\cos\varphi = < q_0, q_1 >$$
$$1 = \alpha^2 + \beta^2 + 2\alpha\beta < q_0, q_1 >$$
$$\cos(u\varphi) = \alpha + \beta < q_0, q_1 >$$

$$1 = \alpha^2 + \beta^2 + 2\alpha\beta\cos\varphi$$
$$\cos(u\varphi) = \alpha + \beta\cos\varphi$$

$$\beta^2\cos^2\varphi + \beta\cos\varphi(1 - 2\cos u\varphi) - 1 = 0$$

$$\beta(u) = \frac{\sin u\varphi}{\sin\varphi} \qquad \alpha(u) = \frac{\sin(1-u)\varphi}{\sin\varphi}$$

Spherical **L**inear Int**erp**olation (**SLERP**)

$$q(u) = \frac{\sin(1-u)\varphi}{\sin\varphi} q_0 + \frac{\sin(u\varphi)}{\sin\varphi} q_1 \qquad u \in [0,1]$$

Equivalent to: $q(u) = q_0(q_0^{-1}q_1)^u$

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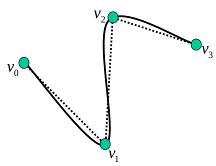
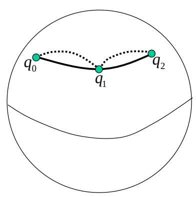
Interpolating Multiple Orientations

- As with linear interpolation between multiple positions, spherical linear interpolation between multiple orientations generates discontinuities in the derivative at the interpolated points.
- This may be solved by more elaborate interpolation schemes, which are not shortest path, similarly to splines for positions.





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