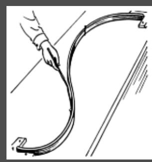
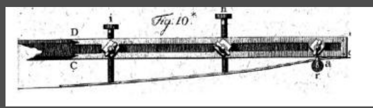
CS5620
Intro to Computer Graphics

Geometric Modeling II

Geometric Modeling
Part II

Physical Splines
Curve design pre-computers





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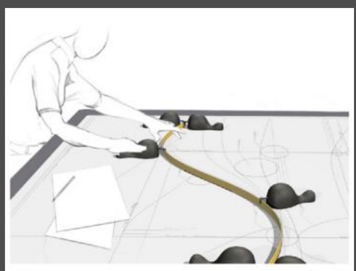

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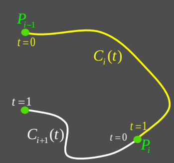
Cubic Splines

- Standard spline input – set of points $\{P_i\}_{i=0,n}$
  - No derivatives' specified as input
- Interpolate by n cubic segments ($4n$ DOF):
  - Force C^1 and C^2 continuity at points
  - Solve $4n$ linear equations in $4n$ unknowns

Interpolation ($2n$ equations):
$$C_i(0) = P_{i-1} \quad C_i(1) = P_i \quad i = 1,..,n$$
C^1 continuity constraints ($n-1$ equations):
$$C_i'(1) = C_{i+1}'(0) \qquad i = 1,..,n-1$$
C^2 continuity constraints ($n-1$ equations):
$$C_i''(1) = C_{i+1}''(0) \qquad i = 1,..,n-1$$



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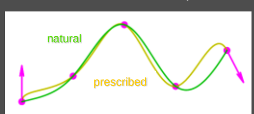
Cubic Splines

- Have two degrees of freedom left (to reach $4n$ DOF)
- Options
  - Natural end conditions: $C_1''(0) = 0,\ C_n''(1) = 0$
  - Complete end conditions: $C_1'(0) = 0,\ C_n'(1) = 0$
  - Prescribed end conditions (derivatives available at the ends): $C_1'(0) = T_0,\ C_n'(1) = T_n$
  - Periodic end conditions $C_1'(0) = C_n'(1),\ C_1''(0) = C_n''(1),$



- Question: What parts of $C(t)$ are affected as a result of a change in P_i ?

demo

Basis functions should be local

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Parameterization

- The assumption $t \in [0,1]$ (uniform parameterization) is arbitrary
  - Implicitly implies same curve "length" for each segment
- Not natural if points are not equally spaced
- One alternative - chord-length parameterization:

$$\text{Denote } d_i = \sqrt{(P_{i-1}^x - P_i^x)^2 + (P_{i-1}^y - P_i^y)^2}$$
$$\text{For the } i\text{'th segment}: t \in [0, d_i].$$

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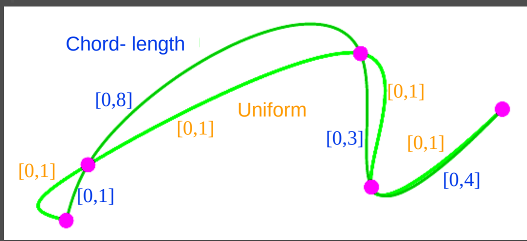
Parameterization



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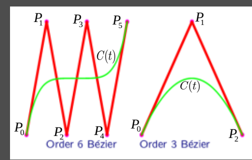
Bezier Curves

- Bezier curve is an ***approximation*** of given control points
- Denote by $C(t)$: $t \in [0,1]$
- Bezier curve of degree n is defined over $n+1$ control points $\{P_i\}_{i=0,n}$



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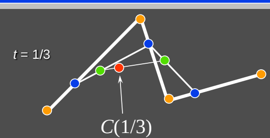
De Casteljau Construction

Select $t \in [0,1]$ value.

$$\begin{aligned}
&For\ i := 0\ to\ n\ do\ P_i^{[0]}(t) := P_i;\\
&For\ j := 1\ to\ n\ do\\
&\quad For\ i := j\ to\ n\ do\\
&\qquad P_i^{[j]}(t) := (1-t)P_{i-1}^{[j-1]}(t) + tP_i^{[j-1]}(t);\\
&C(t) := P_n^{[n]}(t);
\end{aligned}$$



demo

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Algebraic Form of Bezier Curves

Bezier curve for set of control points $\{P_i\}_{i=0,n}$:

$$C(t) = \sum_{i=0}^{n} P_i B_i^n(t) = \sum_{i=0}^{n} P_i \binom{n}{i} (1-t)^{n-i} t^i$$

$\{B_i^n(t)\}_{i=0,n}$ = *Bernstein* basis of polynomials of degree n

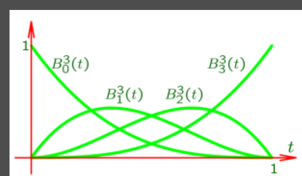
Cubic case:

$B_0^3(t) = (1-t)^3$

$B_1^3(t) = 3(1-t)^2 t$

$B_2^3(t) = 3(1-t)t^2$

$B_3^3(t) = t^3$



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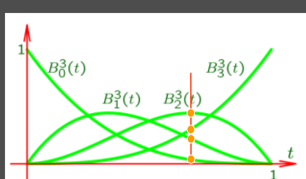
Algebraic Form of Bezier Curves

$C(t) = \sum_{i=0}^{n} P_i \binom{n}{i} (1-t)^{n-i} t^i$

- $\sum_{i=0}^{n} B_i^n(t) = 1, \forall t \in [0,1]$
  - why?
- Curve is linear combination of basis functions
- Curve is convex combination of control points



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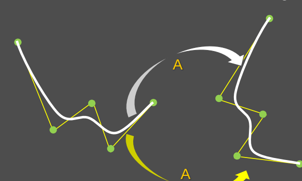
Properties of Bezier Curves

$C(t) = \sum_{i=0}^{n} P_i \binom{n}{i} (1-t)^{n-i} t^i$

- $C(t)$ is polynomial of degree n
- $C(t) \in CH(P_0,\dots,P_n)$
- $C(0) = P_0$ and $C(1) = P_n$
- $C'(t)$ is a Bezier curve of one degree less
- $C'(0) = n(P_1-P_0)$ and $C'(1) = n(P_n-P_{n-1})$
- $C(t)$ is *affine invariant* and *variation diminishing*



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Properties of Bezier Curves

- Questions:
  - What is the shape of Bezier curves whose control points lie on one line?
  - How can one connect two Bezier curves with C^0 continuity? C^1? C^2?

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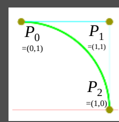
Drawbacks of Bezier Curves

- Degree corresponds to number of control points
  - Global support: change in one control point affects the entire curve
  - For large sets of points – curve deviates far from the points
- Cannot represent conics exactly. Most noticeably circles
  - Can be resolved by introducing a more powerful representation of *rational curves*.
    - For example, a 90 degrees arc as a rational Bezier curve:



$$C(t) = \frac{w_0 P_0 B_0^2(t) + w_1 P_1 B_1^2(t) + w_2 P_2 B_2^2(t)}{w_0 B_0^2(t) + w_1 B_1^2(t) + w_2 B_2^2(t)}$$

where $\frac{w_0 w_2}{w_1^2} = 2.$

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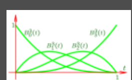
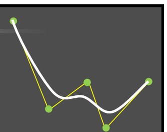
Recap

Bernstein basis functions:





$$B_i^n(t) = \binom{n}{i}(1-t)^{n-i}t^i \qquad \forall t \in [0,1],\ \forall i \quad B_i^n(t) > 0, \quad \sum_{i=0}^{n} B_i^n(t) = 1$$

Bezier curve is *linear* combination of basis functions:

$$C(t) = \sum_{i=0}^{n} P_i B_i^n(t)$$

Bezier curve is *convex* combination of control points (combination depends on *t*):

$$C(t) = \sum_{i=0}^{n} P_i B_i^n(t)$$

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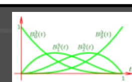
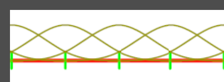
B-Spline Curves



- Idea: Generate basis of functions with *local support*



$$C(t) = \sum_{i=0}^{n-1} P_i N_i(t)$$

- For each parameter value only a finite set of basis functions is non-zero
- The parametric domain is partitioned into sections at integer parameter values (called *knots*).

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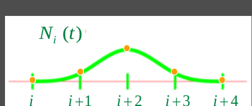
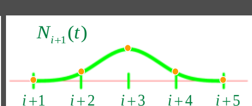
Cubic B-Spline Basis

$$C(t) = \sum_{i=0}^{n-1} P_i N_i(t), \qquad t \in [3,n)$$

$$N_i(t) = \begin{cases} r^3/6 & r = t-i & t \in [i,i+1) \\ (-3r^3+3r^2+3r+1)/6 & r = t-(i+1) & t \in [i+1,i+2) \\ (3r^3-6r^2+4)/6 & r = t-(i+2) & t \in [i+2,i+3) \\ (1-r)^3/6 & r = t-(i+3) & t \in [i+3,i+4) \end{cases}$$





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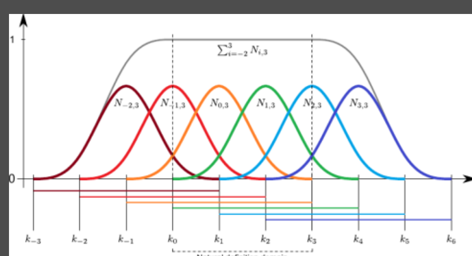
Cubic B-Spline Basis



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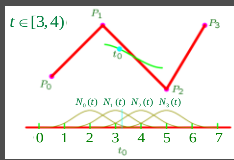
Cubic B-Spline Basis

- For any $t \in [3, n]$: $\sum_{i=0}^{n-1} N_i(t) = 1$
- For any $t \in [3, n]$ at most four basis functions are non zero
- Any point on a cubic B-Spline is a convex combination of at most *four* control points

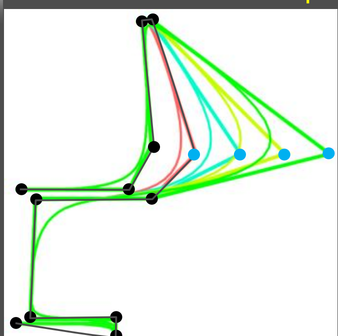
$$C(t) = \sum_{i=0}^{n-1} P_i N_i(t)$$



Boundary Conditions for Cubic B-Spline Curves

- B-Splines do not interpolate control points
  - in particular, the uniform cubic B-spline curves do not interpolate the end points of the curve.
- Ways to force endpoint interpolation:
  - Let $P_0 = P_1 = P_2$ (same for other end)
  - Add a new control point (same for other end) $P_{-1} = 2P_0 - P_1$ and a new basis function $N_{-1}(t)$.

Local Control of B-spline Curves



Control point P_i affects $C(t)$ only for $t \in (i, i+4)$

demo

Properties of B-Spline Curves

- $C(t) = \sum_{i=0}^{n-1} P_i N_i(t), \quad t \in [3, n)$
- For n control points, $C(t)$ is a piecewise polynomial of degree 3, defined over $t \in [3, n)$
- $C(t) \in \bigcup_{i=0}^{n-4} CH(P_i, \ldots, P_{i+3})$
- $C(t)$ is *affine invariant* and *variation diminishing*
- Questions:
  - What is $C(i)$ equal to?
  - What is $C'(i)$ equal to?
  - What is the continuity of $C(t)$? Prove !

From Curves to Surfaces

- A curve is expressed as inner product of coefficients P_i and basis functions

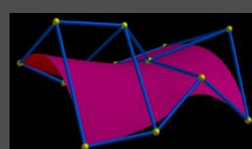
$$C(u) = \sum_{i=0}^{n} P_i B_i(u)$$

- Treat surface as a *curve of curves*. Also known as *tensor product surfaces*
- Assume P_i is not constant, but are functions of a second, new parameter v:

$$P_i(v) = \sum_{j=0}^{m} Q_{ij} B_j(v)$$

From Curves to Surfaces (cont'd)

$$C(u) = \sum_{i=0}^{n} P_i B_i(u) = \sum_{i=0}^{n} \left(\sum_{j=0}^{m} Q_{ij} B_j(v) \right) B_i(u) = \sum_{i=0}^{n} \sum_{j=0}^{m} Q_{ij} B_j(v) B_i(u)$$



$$S(u,v) = \sum_{i=0}^{n} \sum_{j=0}^{m} Q_{ij} B_j(v) B_i(u)$$

BezPatch

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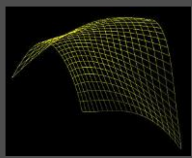
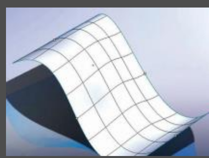
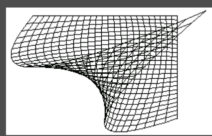
Isoparametric Curves

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Surface Constructors

- Construction of the geometry is a first stage in any *image synthesis* process
- Use a set of high level, simple and intuitive, surface constructors:
  - Bilinear patch
  - Ruled surface
  - Boolean sum
  - Surface of revolution
  - Extrusion surface
  - Swept surface

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Bilinear Patches

- Bilinear interpolation of 4 3D points - 2D analog of 1D linear interpolation between 2 points in the plane
- Given P_{00}, P_{01}, P_{10}, P_{11} the bilinear surface for $u,v \in [0,1]$ is:

$$P(u,v) = (1-u)(1-v)P_{00} + (1-u)vP_{01} + u(1-v)P_{10} + uvP_{11}$$

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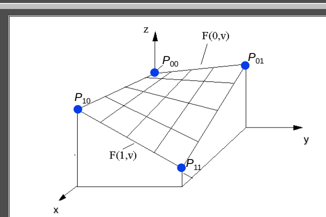
$$P(u,v) = (1-u)(1-v)P_{00} + (1-u)vP_{01} + u(1-v)P_{10} + uvP_{11}$$



Questions:
What does an isoparametric curve of a bilinear patch look like ?
Can you represent the bilinear patch as a Bezier surface ?
When is a bilinear patch planar ?

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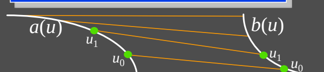
Ruled Surfaces

- Given two curves $a(t)$ and $b(t)$, the corresponding ruled surface between them is:

$$S(u,v) = v\,a(u) + (1-v)b(u)$$



  - The corresponding points on $a(u)$ and $b(u)$ are connected by straight lines
- Questions:
  - When is a ruled surface a bilinear patch?
  - When is a bilinear patch a ruled surface?

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Ruled Surfaces


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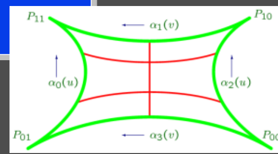
Bridge of Strings

Boolean Sum

- Given four connected curves α_i $i=1,2,3,4$, Boolean sum $S(u,v)$ fills the interior.

$$P(u,v)=(1-u)(1-v)P_{00}+(1-u)vP_{01}+u(1-v)P_{10}+uvP_{11}$$
$$S_1(u,v)=v\alpha_0(u)+(1-v)\alpha_2(u)$$
$$S_2(u,v)=u\alpha_1(v)+(1-u)\alpha_3(v)$$



Then

$$S(u,v)=S_1(u,v)+S_2(u,v)-P(u,v)$$

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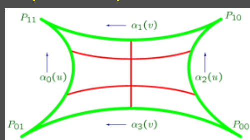
Boolean Sum (cont'd)

- $S(u,v)$ interpolates the four α_i along its boundaries.
- For example, consider the $u=0$ boundary:



$$\begin{aligned}S(0,v)&\\&=S_1(0,v)+S_2(0,v)-P(0,v)\\&=v\alpha_0(0)+(1-v)\alpha_2(0)+0\alpha_1(v)+1\alpha_3(v)-(1-v)P_{00}-vP_{01}\\&=vP_{01}+(1-v)P_{00}+\alpha_3(v)-(1-v)P_{00}-vP_{01}\\&=\alpha_3(v)\end{aligned}$$

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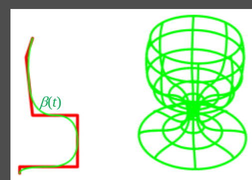
Surface of Revolution

- Rotate a, usually planar, curve around an axis
- Consider curve $\beta(t)=(\beta_x(t),0,\beta_z(t))$ and let Z be the axis of revolution.



$$x(u,v)=\beta_x(u)\cos(v),$$
$$y(u,v)=\beta_x(u)\sin(v),$$
$$z(u,v)=\beta_z(u).$$

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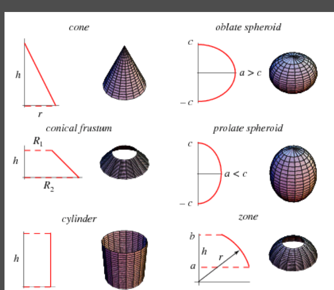
Surfaces of Revolution



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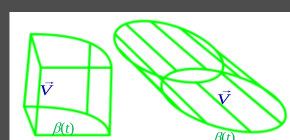
Extruded Surface

- Extrusion of a, usually planar, curve along a linear segment.
- Given curve $\beta(t)$ and vector $\vec{V}$



$$S(u,v)=\beta(u)+v\vec{V}$$

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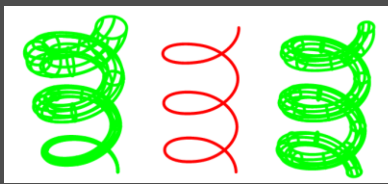
Sweeped Surface

- Rigid motion of one (cross section) curve along another (axis) curve:



- The cross section may change as it is swept

Question: Is an extrusion a special case of a sweep? a surface of revolution?

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