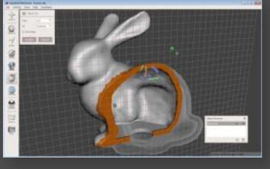
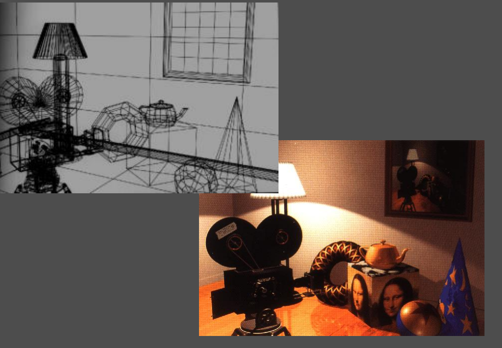
Geometric Modeling
Part I





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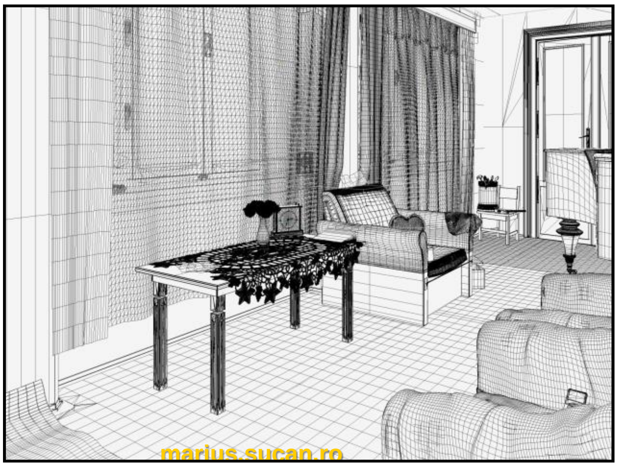

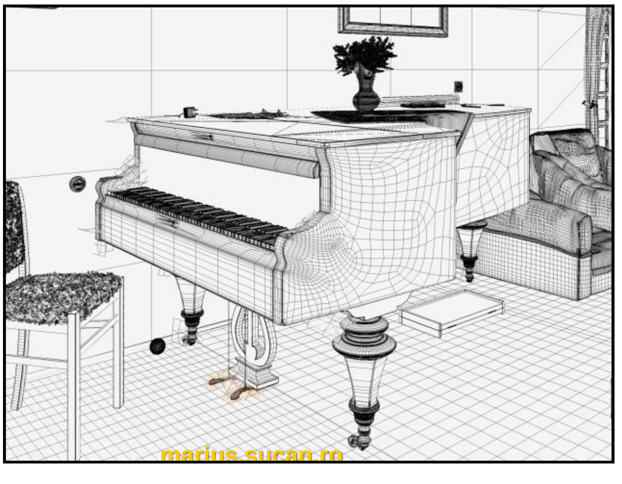

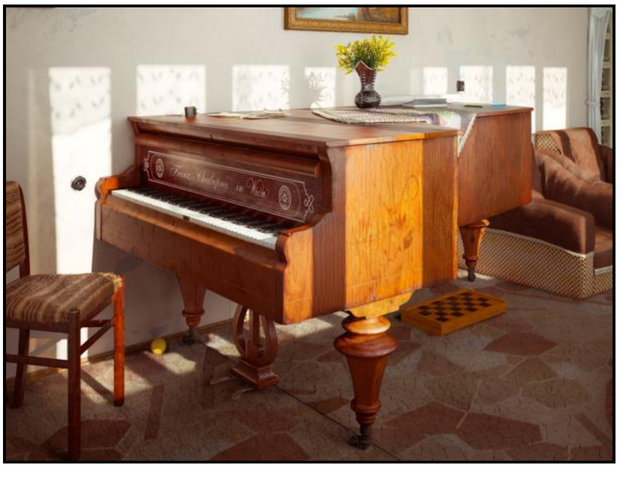

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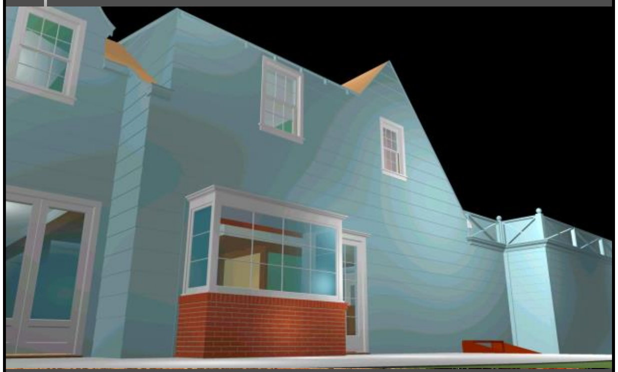

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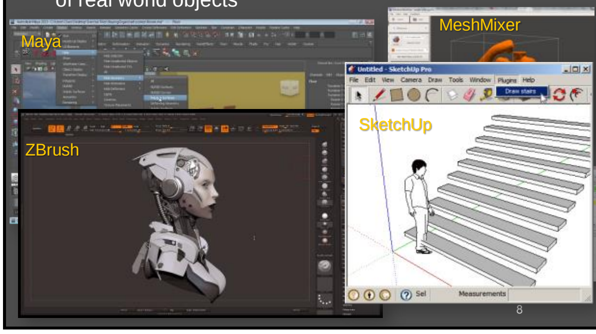
Objective

Methods and algorithms to mathematically model shape of real world objects



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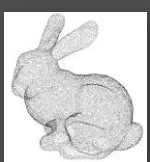
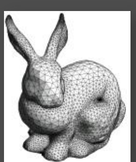
Shape Representations



points



surface

volume

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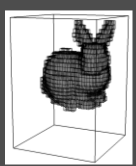
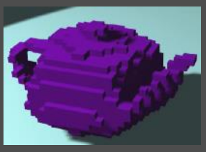
Volumetric Representation

Voxel-based





Advantages: simple and robust Boolean operations, in/out tests, can represent and model the *interior* of the object.

Disadvantages: memory consuming, non-smooth, difficult to manipulate.

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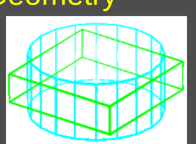
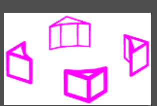
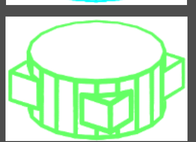
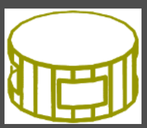
Constructive Solid Geometry

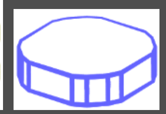
- Use set of volumetric primitives
  - Box, sphere, cylinder, cone, etc…
- For constructing complex objects use Boolean operations
  - Union
  - Intersection
  - Subtraction
  - Complement











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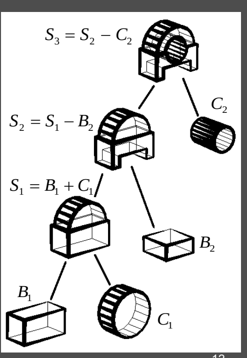
CSG Trees

- Operations performed recursively
- Final object stored as sequence (tree) of operations on primitives
- Common in CAD packages –
  - mechanical parts fit well into primitive based framework
- Can be extended with free-form primitives

Demo



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Surface Representation

- Explicit form: $z = z(x, y)$
- Implicit form: $f(x, y, z) = 0$
- Parametric form: $S(u, v) = [x(u, v), y(u, v), z(u, v)]$

Explicit is a special case of implicit and parametric form

- Example – origin centered sphere of radius R:

Explicit:
$z(x,y) = +\sqrt{R^2 - x^2 - y^2} \cup z = -\sqrt{R^2 - x^2 - y^2}$
Implicit:
$x^2 + y^2 + z^2 - R^2 = 0$
Parametric:
$(x,y,z) = (R\cos\theta\cos\psi, R\sin\theta\cos\psi, R\sin\psi), \theta \in [0,2\pi], \psi \in [-\frac{\pi}{2}, \frac{\pi}{2}]$

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Curve Design

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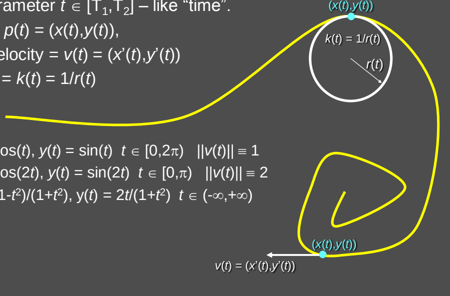
Parametric Curves

- Analogous to trajectory of particle in space.
- Single parameter $t \in [T_1, T_2]$ – like "time".
- position = $p(t) = (x(t), y(t))$,
  tangent velocity = $v(t) = (x'(t), y'(t))$
  curvature = $k(t) = 1/r(t)$
- Circle:
  - $x(t) = \cos(t)$, $y(t) = \sin(t)$ $t \in [0,2\pi)$ $||v(t)|| \equiv 1$
  - $x(t) = \cos(2t)$, $y(t) = \sin(2t)$ $t \in [0,\pi)$ $||v(t)|| \equiv 2$
  - $x(t) = (1-t^2)/(1+t^2)$, $y(t) = 2t/(1+t^2)$ $t \in (-\infty,+\infty)$



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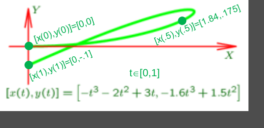
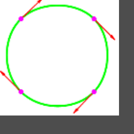
From Functions to Curves

Fit function independently for $x(t)$ and $y(t)$ to obtain $C(t)$





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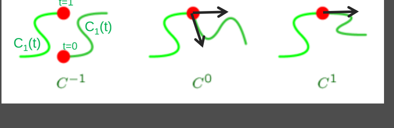
Mathematical Continuity

- $C_1(t)$ & $C_2(t)$, $t \in [0,1]$ - parametric curves
- Level of continuity of the curves at $C_1(1)$ and $C_2(0)$ is:
  - C^{-1}: $C_1(1) \neq C_2(0)$ (discontinuous)
  - C^0: $C_1(1) = C_2(0)$ (positional continuity)
  - C^k, $k > 0$: continuous up to k^{th} derivative

$$C_1^{(j)}(1) = C_2^{(j)}(0), \quad 0 \le j \le k$$



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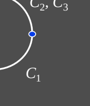
Geometric Continuity

- Mathematical continuity is sometimes too strong
- May be relaxed to geometric continuity
  - G^k, $k \le 0$: Same as C^k
  - G^k, $k = 1$: $C'_1(1) = \alpha C'_2(0)$
  - G^k, $k \ge 0$: There is a reparameterization of $C_1(t)$ & $C_2(t)$, where the two are C^k
- E.g.
  - $C_1(t) = [\cos(t), \sin(t)]$, $t \in [-\pi/2, 0]$
    $C_2(t) = [\cos(t), \sin(t)]$, $t \in [0, \pi/2]$
    $C_3(t) = [\cos(2t), \sin(2t)]$, $t \in [0, \pi/4]$
  - $C_1(t)$ & $C_2(t)$ are C^1 (& G^1) continuous
  - $C_1(t)$ & $C_3(t)$ are G^1 continuous (not C^1)



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Polynomial Curve Fitting

Rules of the game

- Goal:
  - Using a small set of ***control points*** the user can control a curve
- Setup:
  - Looking for a curve $P(t) = (x(t), y(t))$
  - Define $P(t) = \sum_i c_i B_i(t)$, where $B_i(t)$ is a polynomial
  - Use control points information (location, derivatives) to compute the coefficients c_i
    - For example: $P(t_j) = P_j$, where P_j is the control point
- Choices:
  - Which basis functions?
    - What properties should they have?
    - Required curve properties → required basis properties
  - What information from points?

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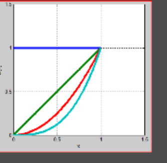
Cubic Monomial Basis

- Basis for cubic polynomials on [0,1]:

$$B = \{M_0(t), M_1(t), M_2(t), M_3(t)\} = \{1, t, t^2, t^3\}$$



$$3t^3 + 4t^2 - 7t - 1 = 3M_3(t) + 2M_2(t) - 7M_1(t) - M_0(t)$$

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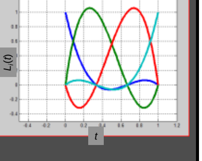
Cubic Lagrange Basis

- Basis for cubic polynomials on [0,1]
- Can interpolate any set of 4 given values p_j

$$B = \{L_0(t), L_1(t), L_2(t), L_3(t)\}$$

$$L_i(t) = \prod_{j=0, j\neq i}^{3} \frac{(t - t_j)}{(t_i - t_j)}$$

$p(t_j) = p_j$ ← $L_i(t_j) = \delta_{ij}$



- E.g. given values on 0, 1/3, 2/3, 1:

$$p(t) = p(0)L_0(t) + p(1/3)L_1(t) + p(2/3)L_2(t) + p(1)L_3(t)$$

- Any cubic can be expressed using the Lagrange basis:

$$3t^3 + 4t^2 - 7t - 1 = -L_0(t) - 2.78L_1(t) - 3L_2(t) - L_3(t)$$

demo

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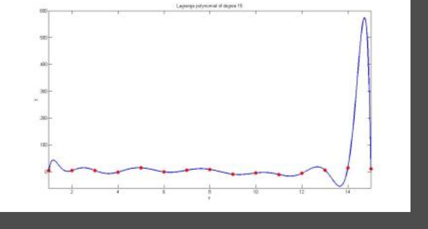

Interpolants based on Lagrange polynomials are not always "nice"

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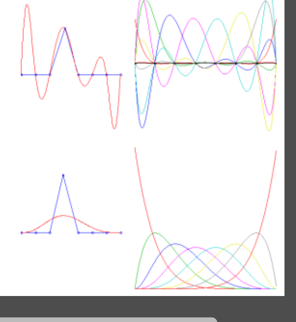
Lagrange basis (order 7)

Non oscillating basis



Basis functions should be non-negative

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Cubic Hermite Basis

- Basis for cubic polynomials on [0,1]

$H_{ij}(t)$: $i, j = 0,1$

- Such that:

	$H(0)$	$H(1)$	$H'(0)$	$H'(1)$
$H_{00}(t)$	1	0	0	0
$H_{01}(t)$	0	1	0	0
$H_{10}(t)$	0	0	1	0
$H_{11}(t)$	0	0	0	1

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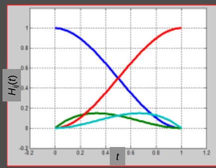
Hermite Cubic Basis

- The four cubics which satisfy these conditions are

$$H_{00}(t) = t^2(2t-3)+1 \qquad H_{01}(t) = -t^2(2t-3)$$
$$H_{10}(t) = t(t-1)^2 \qquad H_{11}(t) = t^2(t-1)$$

- Obtained by solving four linear equations in four unknowns for each basis function
- **Prove**: Hermite cubic polynomials are linearly independent



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Hermite Cubic Basis

	H(0)	H(1)	H '(0)	H '(1)
$H_{00}(t)$	1	0	0	0
$H_{01}(t)$	0	1	0	0
$H_{10}(t)$	0	0	1	0
$H_{11}(t)$	0	0	0	1

- Let's solve for $H_{00}(t)$ as an example.
- $H_{00}(t) = at^3 + bt^2 + ct + d$ should satisfy the following four constraints:

$$H_{00}(0) = 1 = d,$$
$$H_{00}(1) = 0 = a+b+c+d,$$
$$H_{00}'(0) = 0 = c,$$
$$H_{00}'(1) = 0 = 3a+2b+c.$$

- Four linear equations in four unknowns.

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Hermite Cubic Basis (cont'd)

- Let $C(t)$ be a cubic polynomial defined as the linear combination:

$$C(t) = P_0H_{00}(t) + P_1H_{01}(t) + T_0H_{10}(t) + T_1H_{11}(t)$$

- Then $C(0) = P_0$, $C(1) = P_1$, $C'(0) = T_0$, $C'(1) = T_1$
- To generate a curve through P_0 & P_1 with slopes T_0 & T_1, use

$$C(x) = P_0H_{00}(x) + P_1H_{01}(x) + T_0H_{10}(x) + T_1H_{11}(x)$$

demo

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