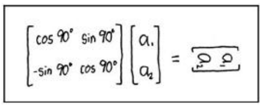
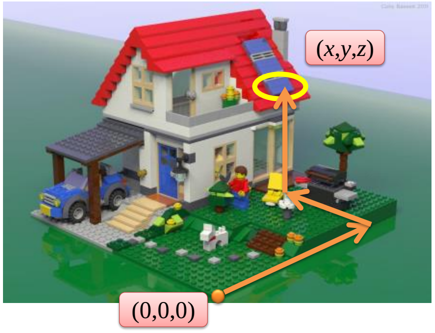
Transformations



Specifying Complex Scenes



2

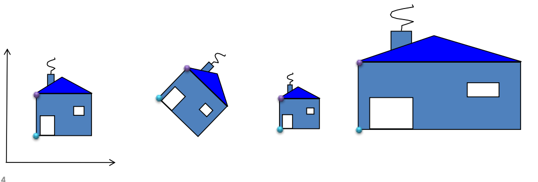
Specifying Complex Scenes

- Absolute position is not very natural
- Need a way to describe "relative" relationship:
  - "The panel is on top of the roof, which is above the house"
- Work "locally" for every object and then combine

3

Transformations

- Transforming an object = transforming all its points
- For a polygon = transforming its vertices

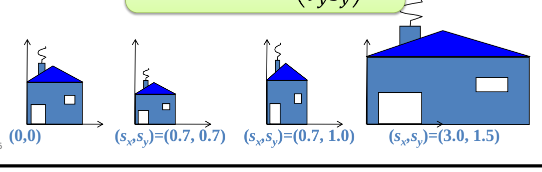


4

Scaling

- $v = \begin{pmatrix} v_x \\ v_y \end{pmatrix}$ – vector in XY plane
- *Scaling* operator S with parameters (s_x, s_y):

$$S^{(s_x,s_y)}(v) = \begin{pmatrix} v_x s_x \\ v_y s_y \end{pmatrix}$$



5

Matrix Form

- Convenient math for representing linear transformations
  - Composition → multiplication
  - Inverse transformation → matrix inverse
  - More on that later…

6

Scaling

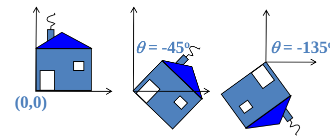
Matrix Form

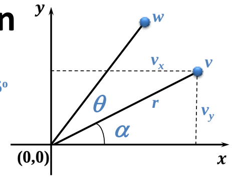
$$S^{s_x,s_y}(v) = \begin{bmatrix} s_x & 0 \\ 0 & s_y \end{bmatrix} \begin{pmatrix} v_x \\ v_y \end{pmatrix} = \begin{pmatrix} v_x s_x \\ v_y s_y \end{pmatrix}$$

- Independent in x and y
  - Non-uniform scaling is allowed

7

Rotation





- Polar form:

$$v = \begin{pmatrix} v_x \\ v_y \end{pmatrix} = \begin{pmatrix} r\cos\alpha \\ r\sin\alpha \end{pmatrix}$$

- Rotate v anti-clockwise by θ to w:

$$R^\theta(v) = \begin{pmatrix} w_x \\ w_y \end{pmatrix} = \begin{pmatrix} r\cos(\alpha+\theta) \\ r\sin(\alpha+\theta) \end{pmatrix} = \begin{pmatrix} v_x\cos\theta - v_y\sin\theta \\ v_x\sin\theta + v_y\cos\theta \end{pmatrix}$$

8

Rotation

- Matrix form:

$$R^\theta(v) = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} v$$

- *Rotation* operator R with parameter θ:

$$R^\theta = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

9

$R^\theta = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$

Rotation Properties

- R^θ is orthogonal

$$\left(R^\theta\right)^{-1} = \left(R^\theta\right)^T$$

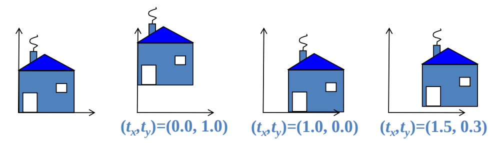
- Rotation by $-\theta$ is

$$R^{-\theta} = \begin{bmatrix} \cos(-\theta) & -\sin(-\theta) \\ \sin(-\theta) & \cos(-\theta) \end{bmatrix}$$

$$= \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} = \left(R^\theta\right)^{-1}$$

10

Translation



- Translation operator T with parameters (t_x, t_y):

$$T^{t_x,t_y}(v) = \begin{pmatrix} v_x + t_x \\ v_y + t_y \end{pmatrix}$$

- Can we express T in a matrix form?

11

Homogeneous Coordinates in R^2

- Points of the form

$$v^h = \begin{pmatrix} v_x^h \\ v_y^h \\ v_w^h \end{pmatrix}$$

- Where we identify

$$\begin{pmatrix} v_x^h \\ v_y^h \\ v_w^h \end{pmatrix} \equiv \begin{pmatrix} cv_x^h \\ cv_y^h \\ cv_w^h \end{pmatrix}$$

for all nonzero c

12

Conversion Formulae

- From Euclidean to homogeneous

$$v = \begin{pmatrix} v_x^h \\ v_y^h \end{pmatrix} \rightarrow v^h = \begin{pmatrix} v_x^h \\ v_y^h \\ 1 \end{pmatrix}$$

- From homogeneous to Euclidean

$$v^h = \begin{pmatrix} v_x^h \\ v_y^h \\ v_w^h \end{pmatrix} \rightarrow v = \begin{pmatrix} \frac{v_x^h}{v_w^h} \\ \frac{v_y^h}{v_w^h} \end{pmatrix}$$

13

Example

- In homogeneous coordinates

$$\begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 4 \\ 4 \\ 2 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 0.5 \end{pmatrix}$$

14

Translation using Homogeneous Coordinates

$$T^{t_x,t_y}(v^h) = \begin{bmatrix} 1 & 0 & t_x \\ 0 & 1 & t_y \\ 0 & 0 & 1 \end{bmatrix} \begin{pmatrix} v_x \\ v_y \\ 1 \end{pmatrix} =$$

$$= \begin{pmatrix} v_x + t_x \\ v_y + t_y \\ 1 \end{pmatrix}$$

15

Matrix Form

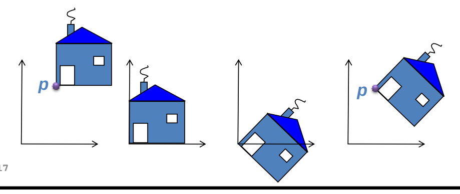
Why bother expressing **transformations** in **matrix** form?

16

Transformation Composition

What operation rotates by θ around $p = \begin{pmatrix} p_x \\ p_y \end{pmatrix}$?

- Translate p to origin
- Rotate around origin by θ
- Translate back



17

Transformation Composition

- M_1 performs one transformation
- M_2 performs a second transformation

$$M_2(M_1 v) = M_2 M_1 v = (M_2 M_1) v$$

- $M_2 M_1$ performs the composed transformation

18

Transformation Composition

$$T^{(p_x,p_y)}R^{\theta}T^{(-p_x,-p_y)}v =$$

$$\begin{bmatrix}1 & 0 & p_x\\0 & 1 & p_y\\0 & 0 & 1\end{bmatrix}\begin{bmatrix}\cos\theta & -\sin\theta & 0\\\sin\theta & \cos\theta & 0\\0 & 0 & 1\end{bmatrix}\begin{bmatrix}1 & 0 & -p_x\\0 & 1 & -p_y\\0 & 0 & 1\end{bmatrix}v$$

19

Transformation Quiz

- What do these Euclidean transformations do?

$$\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix},\begin{bmatrix}1 & 0\\0 & -1\end{bmatrix},\begin{bmatrix}-1 & 0\\0 & 1\end{bmatrix}$$

$$\begin{bmatrix}1 & 0\\1 & 1\end{bmatrix},\begin{bmatrix}1 & a\\0 & 1\end{bmatrix}$$

20

Transformation Quiz

- And these homogeneous ones?

$$\begin{bmatrix}1 & & \\ & 1 & \\ & & -1\end{bmatrix}$$

$$\begin{bmatrix}1 & & \\ & 1 & \\ & & 0.5\end{bmatrix}$$

21

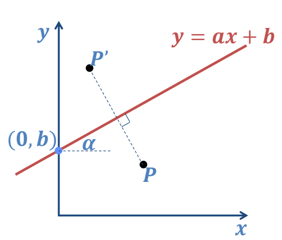
Transformation Quiz

- Can one rotate in the plane by reflection?
- How can one reflect through an arbitrary line in the plane?

22

Arbitrary Reflection

- Shift by $(0,-b)$
- Rotate by $-\alpha$
- Reflect through x
- Rotate by α
- Shift by $(0,b)$



$$T^{(0,b)}R^{\alpha}Ref_xR^{-\alpha}T^{(0,-b)}$$

23

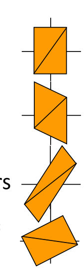
Rotate by Shear

- Shear

$$\begin{bmatrix}1 & 0\\1 & 1\end{bmatrix},\begin{bmatrix}1 & 1\\0 & 1\end{bmatrix},\begin{bmatrix}1 & a\\0 & 1\end{bmatrix},\begin{bmatrix}1 & 0\\a & 1\end{bmatrix}$$

- Rotation by $0 \le \theta \le \frac{\pi}{2} \equiv$ composition of 3 shears

$$\begin{pmatrix}\cos\theta & -\sin\theta\\\sin\theta & \cos\theta\end{pmatrix} = \begin{bmatrix}1 & x\\0 & 1\end{bmatrix}\begin{bmatrix}1 & 0\\y & 1\end{bmatrix}\begin{bmatrix}1 & z\\0 & 1\end{bmatrix} = \begin{bmatrix}1+xy & x\\y & 1\end{bmatrix}\begin{bmatrix}1 & z\\0 & 1\end{bmatrix} = \begin{bmatrix}1+xy & z+xyz+x\\y & yz+1\end{bmatrix}$$



24

Rotate by Shear

$$\begin{bmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{bmatrix} = \begin{bmatrix}1+xy & z+xyz+x\\ y & yz+1\end{bmatrix}$$

- Solve for x, y, z:

$$\cos\theta = 1 + xy = yz + 1$$
$$\sin\theta = y = -(z + xyz + x)$$

- Solution:

$$y = \sin\theta\,,\ x = z = -\tan(\theta/2)$$

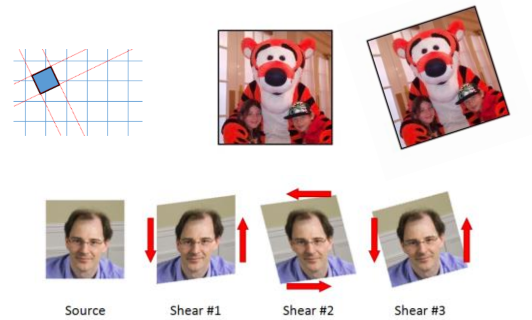
$$\begin{bmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{bmatrix} = \begin{bmatrix}1 & -\tan\theta/2\\ 0 & 1\end{bmatrix}\begin{bmatrix}1 & 0\\ \sin\theta & 1\end{bmatrix}\begin{bmatrix}1 & -\tan\theta/2\\ 0 & 1\end{bmatrix}$$

- When is this useful?

25

Rotate by Shear

Image Rotation



26

Rotate by Shear

- Can we rotate with two (scaled) shears?

$$\begin{bmatrix}1 & \sin\theta\\ 0 & \cos\theta\end{bmatrix}\begin{bmatrix}\sec\theta & 0\\ -\tan\theta & 1\end{bmatrix} = \begin{bmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{bmatrix}$$

- What happens for $\theta \to \frac{\pi}{2}$?

27

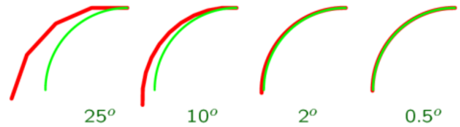
Rotation Approximation

- For small angles $\theta \to 0$ we have:
  $\cos\theta \to 1$ and $\sin\theta \to \theta$ (Taylor expansion of sin/cos)

- Can approximate:

$$\tilde{x} = x\cos\theta - y\sin\theta \cong x - y\theta$$
$$\tilde{y} = x\sin\theta + y\cos\theta \cong x\theta + y$$

Examples (steps of θ):



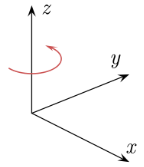
28

3D Transformations

- All 2D transformations extend to 3D
- In homogeneous coordinates:

$$S^{(s_x,s_y,s_z)} = \begin{bmatrix}s_x & & & \\ & s_y & & \\ & & s_z & \\ & & & 1\end{bmatrix},\ T^{(t_x,t_y,t_z)} = \begin{bmatrix}1 & & & t_x\\ & 1 & & t_y\\ & & 1 & t_z\\ & & & 1\end{bmatrix}$$

$$R_z^\theta = \begin{bmatrix}\cos\theta & -\sin\theta & & \\ \sin\theta & \cos\theta & & \\ & & 1 & \\ & & & 1\end{bmatrix}$$



- What is R_x^θ ? R_y^θ ?

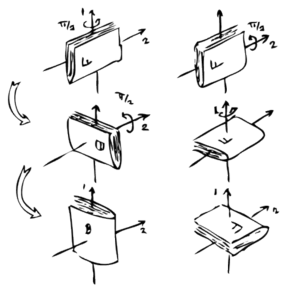
29

3D Transformations

- Questions (commutativity):
  - Scaling: Is $S_1S_2 = S_2S_1$?
  - Translation: Is $T_1T_2 = T_2T_1$?
  - Rotation: Is $R_1R_2 = R_2R_1$?

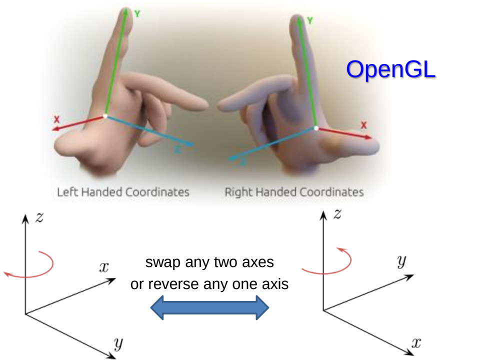
30

Rotation Non-Commutativity



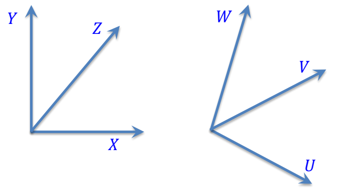
31

3D Coordinate Systems



32

Example: Arbitrary Rotation



Problem:

Given two orthonormal coordinate systems $[X,Y,Z]$ and $[U,V,W]$, find a transformation from one to the other.

33

Arbitrary Rotation

Answer:

Transformation matrix R whose columns are U,V,W:

$$R=\begin{bmatrix}u_x & v_x & w_x\\ u_y & v_y & w_y\\ u_z & v_z & w_z\end{bmatrix}$$

Proof:

$$R(X)=\begin{bmatrix}u_x & v_x & w_x\\ u_y & v_y & w_y\\ u_z & v_z & w_z\end{bmatrix}\begin{pmatrix}1\\0\\0\end{pmatrix}=\begin{pmatrix}u_x\\u_y\\u_z\end{pmatrix}=U$$

Similarly $R(Y)=V$ and $R(Z)=W$

34

Arbitrary Rotation (cont.)

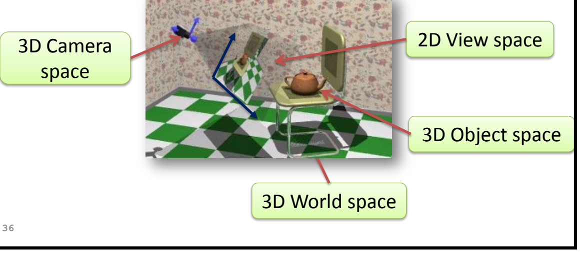
Inverse (=transpose) transformation, R^{-1}, maps $[U,V,W]$ to $[X,Y,Z]$:

$$R^{-1}(U)=\begin{bmatrix}u_x & u_y & u_z\\ v_x & v_y & v_z\\ w_x & w_y & w_z\end{bmatrix}\begin{pmatrix}u_x\\u_y\\u_z\end{pmatrix}=\begin{pmatrix}\|u\|^2\\0\\0\end{pmatrix}=\begin{pmatrix}1\\0\\0\end{pmatrix}=X$$

35

3D Graphics Coordinate Systems

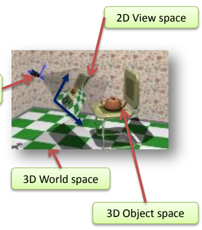
- How can we view (draw) 3D objects on a 2D screen?



36

Viewing Pipeline

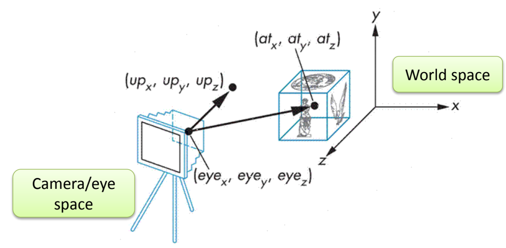
- **object - world**
  positioning the object— modeling transformation
  glTranslate(tx,ty,tz), glScale(sx,sy,sz), glRotate(ang, xa,ya,za)
- **world - camera**
  positioning the camera — viewing transformation
  gluLookAt(cx,cy,cz, ax,ay,az, ux,uy,uz)
- **camera – view**
  taking a picture — projection transformation
  glFrustum(left, right, bottom, top, near,far)
- **view - screen**
  displaying the image — viewport transformation
  glViewport(llx,lly, width,height)



37

Viewing Transformations

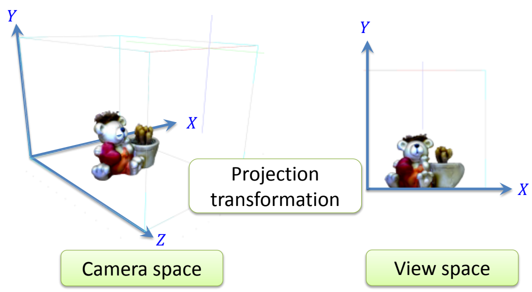
World → Camera/Eye



38

Viewing Transformations

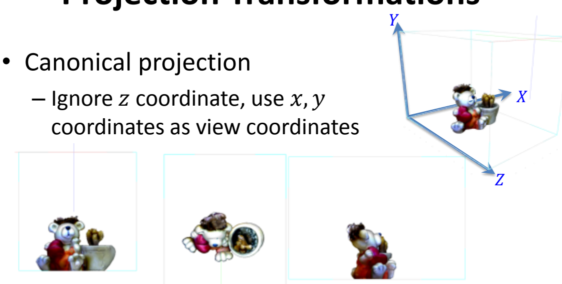
Camera → View



39

Projection Transformations

- Canonical projection
  – Ignore z coordinate, use x, y coordinates as view coordinates

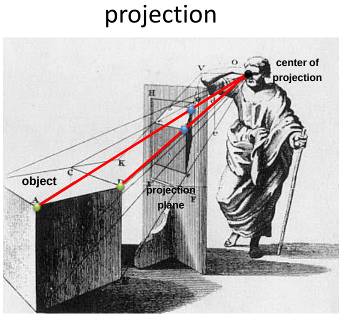


$$\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&0&0\\0&0&0&1\end{bmatrix} \quad \begin{bmatrix}1&0&0&0\\0&0&0&0\\0&0&1&0\\0&0&0&1\end{bmatrix} \quad \begin{bmatrix}0&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{bmatrix}$$

40

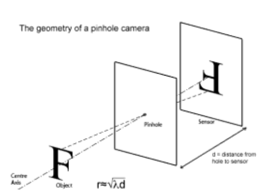
Perspective Projector Lines

A line connecting a point in 3D with its 2D projection



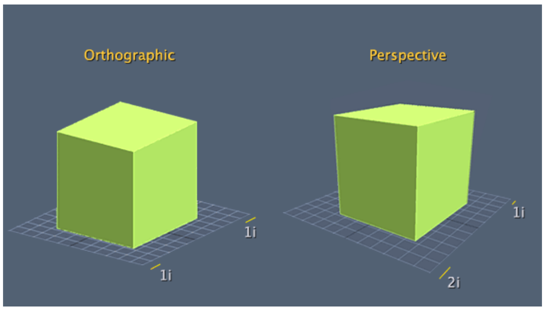
41

Pinhole Camera Model



42

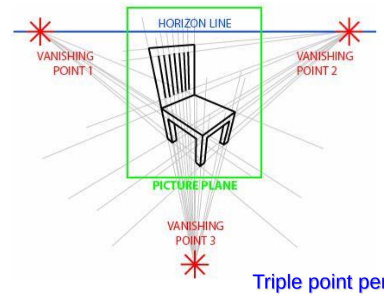
Comparison



43

Vanishing Points

where 3D parallel lines meet in 2D under perspective



Triple point perspective

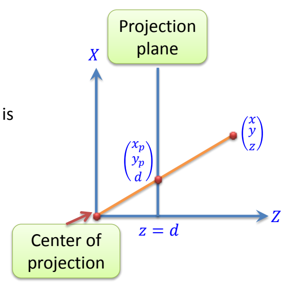
44

Perspective Projection

- Viewing is from *point at a finite distance*
- Without loss of generality:
  - Viewpoint at origin
  - Projection (viewing) plane is $z = d$

Triangle similarity: $\frac{x_p}{d} = \frac{x}{z}, \frac{y_p}{d} = \frac{y}{z}$

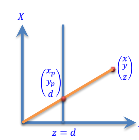
$$x_p = \frac{x}{z/d}, y_p = \frac{y}{z/d}$$



45

Perspective Projection

$\frac{x_p}{d} = \frac{x}{z}, \frac{y_p}{d} = \frac{y}{z}$



- Euclidean coordinates:

$$\begin{pmatrix} x_p \\ y_p \\ d \end{pmatrix} = \begin{pmatrix} \frac{x}{z/d} \\ \frac{y}{z/d} \\ \frac{z}{z/d} \end{pmatrix}$$

- Homogeneous coordinates:

$$\begin{pmatrix} x_p \\ y_p \\ d \\ 1 \end{pmatrix} = \begin{pmatrix} \frac{x}{z/d} \\ \frac{y}{z/d} \\ \frac{z}{z/d} \\ 1 \end{pmatrix} = \begin{pmatrix} x \\ y \\ z \\ z/d \end{pmatrix}$$

- Matrix form:

$$P\begin{pmatrix} x \\ y \\ z \\ 1 \end{pmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & \frac{1}{d} & 0 \end{bmatrix}\begin{pmatrix} x \\ y \\ z \\ 1 \end{pmatrix} = \begin{pmatrix} x \\ y \\ z \\ z/d \end{pmatrix}$$

- P is not injective & singular: $\det(P) = 0$

46

Fun with Perspective

www.pinterest.com/explore/forced-perspective/

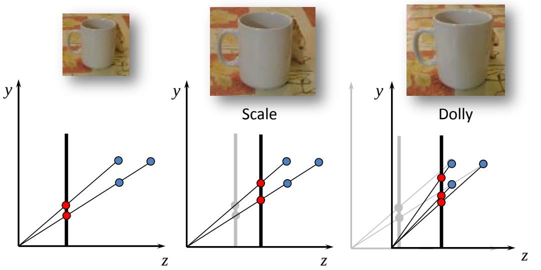
47

JACOBS TECHNION-CORNELL INSTITUTE AT CORNELL TECH

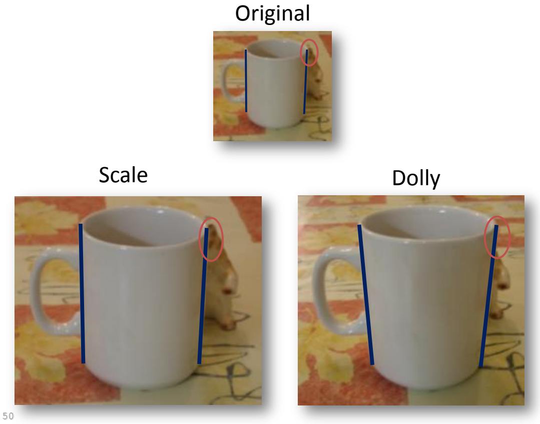
Scale vs. Dolly

What is the difference between:

- Moving the projection plane (**scale, zoom**)
- Moving the viewpoint (center of projection) (**dolly**)



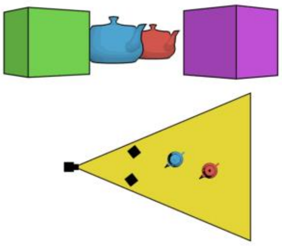
49



50

The Dolly-Zoom Effect
aka “Hitchcock effect”

- Dolly + scale s.t. subject remains same size
- Zoom-in camera while moving away from subject (or the opposite)



vimeo.com/moogaloop.swf?clip_id=84548119

51

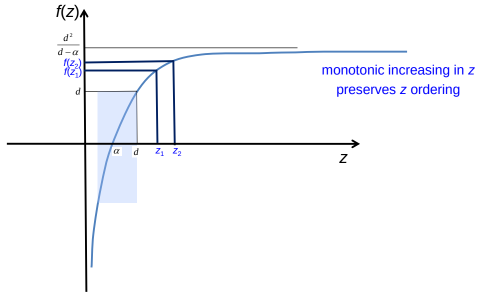
Perspective Warp

$$\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&\frac{d}{d-\alpha}&\frac{-\alpha d}{d-\alpha}\\0&0&1/d&0\end{bmatrix}\begin{pmatrix}x\\y\\z\\1\end{pmatrix}=\begin{pmatrix}x\\y\\\frac{d(z-\alpha)}{d-\alpha}\\z/d\end{pmatrix}$$

$$\begin{pmatrix}x_p\\y_p\\z_p\end{pmatrix}=\begin{pmatrix}\frac{x}{z/d}\\\frac{y}{z/d}\\\frac{d^2(z-\alpha)}{z(d-\alpha)}\end{pmatrix}$$

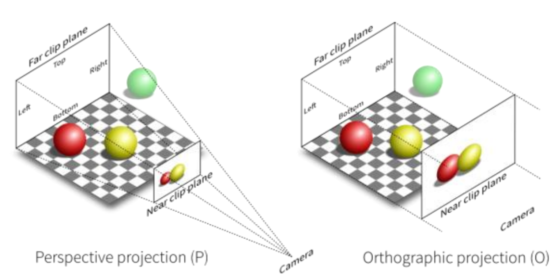
52

$$f(z)=\frac{d^2(z-\alpha)}{z(d-\alpha)}=\frac{d^2}{d-\alpha}\left(1-\frac{\alpha}{z}\right)$$



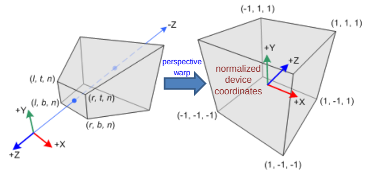
53

View Frustrum



54

Projection Matrix
From View Frustrum



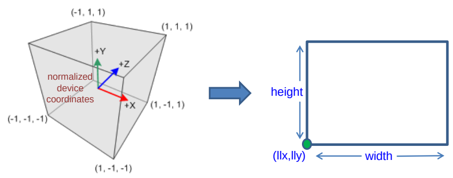
$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \frac{d}{d-\alpha} & \frac{-\alpha d}{d-\alpha} \\ 0 & 0 & 1/d & 0 \end{bmatrix}$$

$$\begin{bmatrix} \frac{2n}{r-l} & 0 & \frac{r+l}{r-l} & 0 \\ 0 & \frac{2n}{t-b} & \frac{t+b}{t-b} & 0 \\ 0 & 0 & \frac{-(f+n)}{f-n} & \frac{-2fn}{f-n} \\ 0 & 0 & -1 & 0 \end{bmatrix}$$

Important special case:
$r = -l, t = -b$

55

Viewport Matrix
From NDC to Viewport



56

Fun with Perspective

57

@Julian Beever

3D Sidewalk Art by Julian Beever

58

@Julian Beever

59

@Julian Beever

60

JACOBS TECHNION-CORNELL INSTITUTE AT CORNELL TECH


WORLD
61


62


63