ECE/ENGRD 2300
Digital Logic & Computer Organization
Spring 2025

Boolean Algebra

Cornell University

Announcements

- Course staff email <ece2300-staff-L@cornell.edu> is active now

- HW 1 will be released on CMS today

- Labs and office hours start next week
  - More logistics will be posted on Ed soon

Review: Digital Abstraction

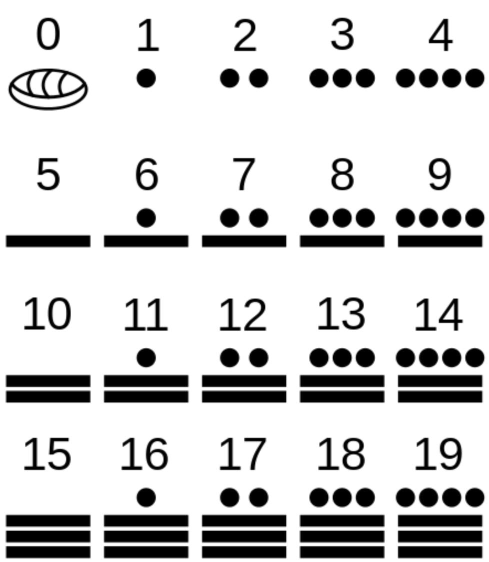
- Typically digital implies a *binary* system (base 2)
  - Only 2 recognized values (logic levels)
    - High / Low
    - True / False
    - On / Off

    …

- Logic levels are particular voltage ranges
  - Don't allow "0" to be easily mistaken for a "1" or vice versa

Fun Facts: Number Bases

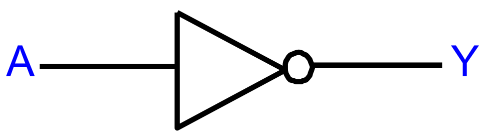
- Digital computers represent numbers in binary
- We count in 10
- Mayans used positional notion based on 20



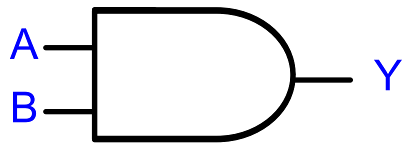
Maya numerals

Review: Logic Gates

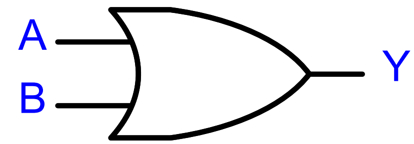
- **Logic gates are functions: take one or more binary inputs and produce a binary output**
  - **Note that the input values are not constants**



NOT Gate



AND Gate



OR Gate

A’

A	Y
0	1
1	0

Truth Table

A•B

A	B	Y
0	0	0
0	1	0
1	0	0
1	1	1

A+B

A	B	Y
0	0	0
0	1	1
1	0	1
1	1	1

Exercise: Truth Table

- Write down the truth table for a three-input function F = f(A,B,C) that outputs
  - F = 1 if (A•B) is <u>equal</u> to (B+C')
  - F = 0 otherwise

- Fill out the blank entries

ABC	F
000	0
001	1
010	0
011	0
100	
101	1
110	
111	1

Boolean Algebra

- Mathematical foundation for analyzing and simplifying digital circuits

- Boolean algebra (George Boole, 1854)
  - Two-valued algebraic system
  - Used to formulate true or false postulations

- Switching algebra (Claude Shannon, 1938)
  - Adopted Boolean algebra for digital circuits in his master's thesis "A Symbolic Analysis of Relay and Switching Circuits"
  - Terms "Boolean algebra" and "switching algebra" are often used interchangeably

Basic “Ingredients”

- Variables that have values of either 1 or 0 (True or False)

- Basic operators are AND, OR, and NOT

Some Important Definitions

- Literal: a single variable or its complement
  - e.g., A (positive literal), A’ (negative literal)
- Product term: AND of (more than one) literals
  - e.g., A’•B
- Sum term: OR of literals
  - e.g., A+B+C’

Operator Precedence

- What does D•A’+B•C mean?

- Operator precedence rules
  1. NOT (highest priority)
  2. AND
  3. OR (lowest)

Some More Definitions

- **Normal term**: Product or sum term in which every variable appears, and exactly once
- **Minterm**: Normal product
  - e.g., (A•B'•C) for a 3-input Boolean function
- **Maxterm**: Normal sum
  - e.g., (A'+B+C')

Minterms & Maxterms

ABC	**Minterm**	name	**Maxterm**	name
000	A’•B’•C’	m_0	A+B+C	M_0
001	A’•B’•C	m_1	A+B+C’	M_1
010	A’•B•C’	m_2	A+B’+C	M_2
011	A’•B•C	m_3	A+B’+C’	M_3
100	A•B’•C’	m_4	A’+B+C	M_4
101	A•B’•C	m_5	A’+B+C’	M_5
110	A•B•C’	m_6	A’+B’+C	M_6
111	A•B•C	m_7	A’+B’+C’	M_7

Each input combination corresponds to a unique minterm/maxterm

Input 0 corresponds to a negative literal in a minterm, 1 to a positive; The opposite applies to maxterms

Canonical Form of a Boolean Function

- A canonical form is a distinct representation for a Boolean function, ensuring that
  - with a fixed ordering of input variables, two equivalent functions have the same form

- Truth table is a canonical form
  - e.g., (A'+B') and (A•B)' have the same truth table
  - 2^n rows required for an n-input function

- Other canonical representations
  - Canonical sum, i.e., canonical sum of products (SOP)
  - Canonical product, i.e., canonical product of sums (POS)
  - Other (more) compact forms exist (outside the scope of 2300)

Canonical Sum of a Boolean Function

- <u>Sum of minterms</u> that correspond to the on-set of a function
  - On-set: the set of input combinations for which the function produces an output of 1

F = 1 if *<u>one</u>* of the input combinations *<u>from on-set</u>* is selected; Otherwise, F=0

(Intuition: F=1 when input "hits" the onset)

$$F = A'\bullet B'\bullet C' + A'\bullet B\bullet C + A\bullet B'\bullet C' + A\bullet B\bullet C$$
$$= m_0+m_3+m_4+m_7$$
$$= \Sigma_{A,B,C}(0,3,4,7)$$

ABC	F
000	1
001	0
010	0
011	1
100	1
101	0
110	0
111	1

Canonical Product of a Boolean Function

- Product of maxterms that correspond to the off-set of a function
  - **Off-set**: the set of input combinations for which the function produces an output of 0

F = 1 if *none* of the input combinations *from off-set* is selected; Otherwise, F=0

(Intuition: F=1 when input "escapes" the offset)

$F = (A+B+C')\bullet(A+B'+C)\bullet(A'+B+C')\bullet(A'+B'+C)$

$= M_1\bullet M_2\bullet M_5\bullet M_6$

$= \Pi_{A,B,C}(1,2,5,6)$

ABC	F
000	1
001	0
010	0
011	1
100	1
101	0
110	0
111	1

Different Forms, Same Function

- The canonical sum and canonical product are distinct (canonical) forms, representing the same function

$$F = \Sigma_{A,B,C}(0,3,4,7) = \Pi_{A,B,C}(1,2,5,6)$$

ABC	F
000	1
001	0
010	0
011	1
100	1
101	0
110	0
111	1

Exercise: Canonical Forms

- Write down the canonical sum & product forms for basic operators NOT, AND, OR

A	F
0	1
1	0

A	B	F
0	0	0
0	1	0
1	0	0
1	1	1

A	B	F
0	0	0
0	1	1
1	0	1
1	1	1

Axioms of Boolean Algebra

- Statements that are assumed true
- Obey the principle of duality
  - Many axioms come in pairs
  - Interchange 1 and 0, AND and OR, still correct

Axioms of Boolean Algebra

- **Binary**

  (A1) $X = 0$ if $X \neq 1$ **(A1')** $X = 1$ if $X \neq 0$

- **Complement**

  (A2) If $X = 0$, then $X' = 1$ **(A2')** If $X = 1$, then $X' = 0$

Axioms of Boolean Algebra

- AND and OR

(A3)	$0 \bullet 0 = 0$	(A3')	$1 + 1 = 1$
(A4)	$1 \bullet 1 = 1$	(A4')	$0 + 0 = 0$
(A5)	$0 \bullet 1 = 1 \bullet 0 = 0$	(A5')	$1 + 0 = 0 + 1 = 1$

- A1-A5 completely define Boolean algebra
  - Everything else derived from these axioms

Single Variable Theorems

• Identity:	(T1)	$X \cdot 1 = X$	(T1')	$X + 0 = X$
• Null Element:	(T2)	$X \cdot 0 = 0$	(T2')	$X + 1 = 1$
• Idempotency:	(T3)	$X \cdot X = X$	(T3')	$X + X = X$
• Involution:	(T4)	$(X')' = X$		
• Complements:	(T5)	$X \cdot X' = 0$	(T5')	$X + X' = 1$

- Can prove by *perfect induction*
  - Show that all possible inputs meet the theorem

Exercise: Proof by Perfect Induction

(T3) $X \bullet X = X$

$X=0 \rightarrow 0 \bullet 0 = 0$

$X=1 \rightarrow 1 \bullet 1 = 1$

(T3') $X + X = X$

$X=0 \rightarrow 0+0=0$

$X=1 \rightarrow 1+1=1$

(T5) $X \bullet X' = 0$

(T5') $X + X' = 1$

Two and Three Variable Theorems

- **Commutativity**

(T6) $X \bullet Y = Y \bullet X$ (T6') $X+Y = Y+X$

- **Associativity**

(T7) $(X \bullet Y) \bullet Z = X \bullet (Y \bullet Z)$ (T7') $(X+Y)+Z = X+(Y+Z)$

- **Distributivity**

(T8) $X \bullet Y+X \bullet Z = X \bullet (Y+Z)$ (T8') $(X+Y) \bullet (X+Z) = X+(Y \bullet Z)$

AND distributes over OR OR distributes over AND

Two and Three Variable Theorems

- **Covering**

  (T9) $X \bullet (X+Y) = X$ (T9') $X + X \bullet Y = X$

- **Combining**

  (T10) $X \bullet Y + X \bullet Y' = X$ (T10') $(X+Y) \bullet (X+Y') = X$

- **Consensus**

  (T11) $X \bullet Y + X' \bullet Z + Y \bullet Z = X \bullet Y + X' \bullet Z$

  (T11') $(X+Y) \bullet (X'+Z) \bullet (Y+Z) = (X+Y) \bullet (X'+Z)$

Exercise

- Prove algebraically: (T9') $X+X \bullet Y = X$

Exercise

- **Prove algebraically: (T9') $X+X \bullet Y = X$**

- **Solution**

  – $X + X \bullet Y \underset{\text{(identity)}}{\overset{T1}{=}} X \bullet 1 + X \bullet Y \underset{\text{(distributivity)}}{\overset{T8}{=}} X \bullet (1+Y) \underset{\text{(null elements)}}{\overset{T2'}{=}} X \bullet 1 \underset{\text{(identity)}}{\overset{T1}{=}} X$

De Morgan's Theorem

- Very important, also known as De Morgan's *Law*

(T12) $(X1 \bullet X2 \bullet \ldots \bullet Xn)' = X1' + X2' + \ldots + Xn'$

(T12') $(X1 + X2 + \ldots + Xn)' = X1' \bullet X2' \bullet \ldots \bullet Xn'$

De Morgan Example

- By DeMorgan's Law

  $(X \bullet Y \bullet Z)' = X' + Y' + Z'$

- Proof by perfect induction

XYZ	$(X \bullet Y \bullet Z)'$	$X'+Y'+Z'$
000	1	1
001	1	1
010	1	1
011	1	1
100	1	1
101	1	1
110	1	1
111	0	0

Next Class

Combinational Logic Minimization
(H&H 2.4-2.7)